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INVESTIGATION OF A CLASS OF FUNDAMENTAL INEQUALITIES IN THE THEORY OF ANALYTIC FUNCTIONS.

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(Authorized translation from the Danish* by T. H. Gronwall.)

TABLE OF CONTENTS.

§ 1. Introduction.

In recent years, a class of function theoretic problems have acquired a steadily increasing importance, viz. problems of the following nature. We consider an otherwise arbitrary analytic function $y = y(x)$ of the complex variable x, and this function y is assumed to be holomorphic inside the circle $|x| < R$; in other words, we consider an arbitrary power series in x with a radius of convergence $\geqq R$. Moreover, we assume regarding this function that $|y| < M$, or $\Re(y) < A$, or similar conditions,† all this for $|x| < R$, and finally that we know the value of the function at a given point inside the circle, usually the center, i.e., $y(0)$ is known. What further general statements may then be made regarding the function, what may be said of its absolute value, of its real or its imaginary part, of its zeros or of regions where there are no zeros, of its derivative, and so forth?

Before presenting my own investigations, I shall review briefly (and to the extent of my knowledge of the literature on the subject) the more or less special results in this direction obtained by those mathematicians who have made substantial contributions to the solution of problems of this nature. To facilitate comparison, I shall *use greatly changed notations throughout*, which are in accord with those I use later on.

* Undersögelser over en Klasse fundamentale Uligheder i de analytiske Funktioners Theori. D. Kgl. Danske Vidensk. Selsk. Skrifter, Naturv. og Mathematisk Afd., ser. 8, vol. 2, p. 199–228 (1916). Essential parts of this paper were communicated earlier in two lectures, one on the occasion of presenting these investigations to the Royal Danish Academy of Sciences at the meeting Nov. 19, 1915, and another, previous to this, before the Copenhagen Mathematical Society Sept. 16, 1915, on reckoning with ordinary complex numbers with applications to function theory.

† For the notations, see § 2.

1. (1869.) In the first place, mention should be made of a proposition due to Schwarz[1] which may be stated as follows: When y, besides being holomorphic and less than M in absolute value inside the circle $|x| < R$, has the property that $y(0) = 0$, then we have for $|x| < R$

$$|y| \leqq \frac{M|x|}{R}.$$

Carathéodory[2] was probably first in emphasizing the great importance of this proposition, and proved it in a very simple and elementary manner. The name "Schwarz' lemma" introduced by this author will be adhered to in the following.

2. (1892.) Toward the discovery of properties of y inside the circle frequently referred to, when a positive upper bound of the real part of the function is given, Hadamard[3] made a contribution which led Borel[4] to the following solution of a problem of this kind:

$$|y| < |\Re y(0)| + |\Im y(0)| + (4A + 2|\Re y(0)|)\frac{|x|}{R - |x|},$$

where A denotes an upper (positive) bound of the real part $\Re(y)$ of y, and $\Im(y)$ is the "imaginary part" of y (or more correctly expressed, the coefficient of i in the purely imaginary part). Schottky[5] improved upon this solution by means of the following formula obtained from the Cauchy integral

$$|y| \leqq |\Im y(0)| + \frac{1}{2\pi}\int_0^{2\pi} |\Re y(re^{i\theta})|\, d\theta \cdot \frac{r + |x|}{r - |x|},$$

where $|x| < r < R$. In accordance with the line of reasoning in the paper quoted in the footnote, this formula gives, with the notations used above

$$|y| \leqq |\Im y(0)| + (2A - \Re y(0))\frac{R + |x|}{R - |x|}.$$

[1] H. A. Schwarz, "Zur Theorie der Abbildung." Programm der eidgenössischen polytechnischen Schule in Zürich für das Schuljahr 1869–70; Gesammelte Mathematische Abhandlungen (Berlin, 1890), vol. 2, p. 108–132 (see p. 110–111).

[2] C. Carathéodory, "Sur quelques généralisations du théorème de M. Picard," Comptes rendus de l'Académie des Sciences, vol. 141, Paris, 1905, p. 1213–1215.

E. Landau, "Ueber den Picardschen Satz," Vierteljahrsschrift der naturforschenden Gesellschaft in Zürich, vol. 51, 1906, p. 252–318 (see p. 271).

C. Carathéodory, "Untersuchungen über die konformen Abbildungen von festen und veränderlichen Gebieten," Mathematische Annalen, vol. 72, 1912, p. 107–144 (see p. 110).

[3] J. Hadamard, "Sur les fonctions entières de la forme $e^{G(x)}$," Comptes rendus, vol. 114, 1892, p. 1053–1055.

[4] E. Borel, "Démonstration élémentaire d'un théorème de M. Picard sur les fonctions entières," Comptes rendus, vol. 122, 1896, p. 1045–1048.

[5] F. Schottky, "Ueber den Picard'schen Satz und die Borel'schen Ungleichungen," Sitzungsberichte der Kgl. Preussischen Akademie der Wissenschaften, 1904, p. 1244–1262 (see p. 1246–1247).

Carathéodory[6] improved this to

$$| y | \leqq | \Im y(0) | + 2A \frac{| x |}{R - | x |} + | \Re y(0) | \frac{R + | x |}{R - | x |},$$

where A is no longer assumed to be positive. This formula is commonly called "Carathéodory's theorem."

3. (1899.) Regarding the zeros of y the author[7] gave a result which may be stated as follows: When y is holomorphic for $| x | < R$, and $| y | < M$, and when $x_1, x_2, \cdots, x_n$ denote those zeros of y which fall inside or on the circumference $| x | = r < R$, we have

$$M > | y(0) | \frac{R^n}{| x_1 x_2 \cdots x_n |}.$$

Here it is evident that y has no zeros for

$$| x | < \frac{| y(0) |}{M} R.$$

An improvement on this proposition is due to Carathéodory and Fejér[8] who, besides giving a new proof, showed that the equality sign will occur in the inequality above when y belongs to a certain class of rational functions of x. (Incidentally I may state that this note together with my own methods for easy reckoning with complex numbers have formed an essential reason for the publication of the present paper.)

A somewhat more general result has been obtained recently by Landau[9] who shows that when η is a constant less than M in absolute value, and $x_1, x_2, \cdots, x_n$ are zeros of the function $y - \eta$ inside $| x | = R$, and η and η denote conjugate numbers, then

$$\left| \frac{M^2 - \bar{\eta} y(0)}{M(y(0) - \eta)} \right| \geqq \frac{R^n}{| x_1 x_2 \cdots x_n |},$$

whence it may be shown that

$$y \neq \eta \qquad \text{for} \qquad | x | < \left| \frac{M(y(0) - \eta)}{M^2 - \bar{\eta} y(0)} \right| R.$$

Moreover, Landau gives a direct proof of the last proposition.

[6] E. Landau, l. c.,[2] p. 277. See also E. Landau, "Beiträge zur analytischen Zahlentheorie," Rendiconti del Circolo Matematico di Palermo, vol. 26, 1908, p. 169–302 (see p. 191–193).

[7] J. L. W. V. Jensen, "Sur un nouvel et important théorème de la théorie des fonctions," Acta Mathematica, vol. 22, 1899, p. 359–364 (see p. 362–363).

The author also presented this formula in a lecture before the Copenhagen Mathematical Society three years earlier. The proof given here was entirely elementary.

[8] C. Carathéodory et L. Fejér, "Remarques sur le théorème de M. Jensen," Comptes rendus, vol. 145, 1907, p. 163–165.

[9] E. Landau, "Ueber eine Aufgabe aus der Funktionentheorie," The Tôhoku Mathematical Journal, vol. 5, 1914, p. 97–116 (see p. 107 and p. 105).

4. (1906.) In regard to $y'(x)$, the derivative of y, the first result is due to Landau (l. c.,[2] p. 305–306) who shows that when $|y| < M$ for $|x| < R$, then

$$|y'(0)| \leqq \frac{M^2 - |y(0)|^2}{RM}.$$

A corresponding result obtained by F. W. Wiener[10] may be written as follows:

$$|y^{(n)}(0)| \leqq n! \frac{M^2 - |y(0)|^2}{R^n M}.$$

5. (1908.) The most comprehensive set of results concerning the problems treated here was given however by Lindelöf;[11] by means of the theory of Green's functions and conformal mapping he sets up a principle which, as he shows at length, gives the solution of a whole set of problems containing those mentioned in 1, 2, and 4 as special cases. To these are added new, elegant and important results, such as an extension of Schwarz' lemma

$$|y| \leqq M \frac{R|y(0)| + M|x|}{RM + |y(0)x|} \qquad \text{when} \qquad |y| < M \text{ for } |x| < R,$$

an elegant extension of Landau's formula above

$$|y'(x)| \leqq \frac{R}{M} \frac{M^2 - |y|^2}{R^2 - |x|^2},$$

and others, not to mention the determination of a region in the circle $|x| < R$ and in the vicinity of an arbitrary point x_0, in which region $y \neq 0$. He does not consider, however, the other problems falling under 3, and he restricts himself, as do most of the other authors, to assuming $y(0)$ as the known value of y instead of making the easy extension to the case when $y_0 = y(x_0)$ is assumed as the known value, x_0 being fixed arbitrarily. In the following, I shall have occasion to mention the results of Lindelöf individually, since they will occur as special cases of other and more general theorems.

6. (1914.) Carathéodory[12] was probably first in giving a formula for

[10] H. Bohr, "A theorem concerning power series," Proceedings of the London Mathematical Society, ser. 2, vol. 13, 1913 (see the lemma used in this paper).

[11] E. Lindelöf, "Mémoire sur certaines inégalités dans la théorie des fonctions monogènes et sur quelques propriétés nouvelles de ces fonctions dans le voisinage d'un point singulier essentiel," Acta Societatis Scientiarum Fennicae, vol. 25, no. 7, p. 1–35.

[12] C. Carathéodory, "Elementarer Beweis für den Fundamentalsatz der konformen Abbildungen," Mathematische Abhandlungen Hermann Amandus Schwarz zu seinem fünfzigjährigen Doktorjubiläum am 6. August 1914 gewidmet, Berlin, 1914, p. 19–41 (see p. 23).

the difference quotient, viz.,

$$\frac{|y - y_0|}{|x - x_0|} \leqq \frac{R\,|M^2 - \bar{y}_0 y|}{M\,|R^2 - \bar{x}_0 x|} < \frac{2MR}{R^2 - |x_0 x|}.$$

Regarding these formulas, we may note that the first inequality gives Lindelöf's formula above when $x \to x_0$; the second is only a crude approximation fully sufficient, however, for the purposes of the paper quoted.

A close examination of these solutions of special problems reveals a lack of unity in treatment which it seems desirable to remove. In the present paper, I have therefore worked these theorems into a connected whole and generalized them considerably, so that my results contain all the previous ones as very special cases, as well as a number of new propositions, no special cases of which were known previously; I have also attempted to do this in the simplest and most elementary manner. I hope that it will appear from the following that it will hardly be possible to reduce the solutions of these problems to a higher degree of simplicity or base them on a smaller number of function theoretic propositions, all of the latter belonging moreover to that part of the elementary theory of functions which is most elegantly treated by means of power series.

It is almost superfluous to state that the celebrated theorems of Picard, or rather the elementary proofs and generalizations due to Borel, Landau and Schottky, are intimately connected with some of our problems, particularly those mentioned in 2. This is apparent moreover from the titles quoted under [2, 4, 5]. It is however beyond the scope of the present paper to consider applications of the theorems presented.

Before proceeding to those rules for reckoning with ordinary complex numbers which form the greater part of the lemmas used in the following (and which could surely find their place in a textbook on elementary algebra), I wish to make a remark on the problems treated and their solutions. In these, the inequalities which together with certain simple assumptions form the basis of the solution, are used to derive other inequalities, frequently altogether different in form. In certain cases, and often in the most important problems, it is possible by an algebraic transformation of the final inequalities to return a fortiori to precisely the initial conditions. To make this clear in the very simplest case, let us consider Schwarz' lemma. From the assumptions, we derive the solution $|y| \leqq M|x|/R$; from this inequality it follows at once that $y(0) = 0$ and that, a fortiori, $|y| < M$, which are exactly the assumptions from which we started; moreover, the solution contains an equality sign which cannot be dispensed with. When these circumstances occur in the solution of a problem, as will be the case continually in the following,

we say that the solution is *complete*. A final remark on inequalities is this: When an inequality $A_1 \leqq B_1$ leads to another $A_2 \leqq B_2$ by a finite number of algebraic operations, and *vice versa*, and when moreover an inequality sign corresponds to an inequality sign, a sign of equality to a sign of equality, these two inequalities are said to be *equivalent*. On the contrary, when the first inequality does not follow from the second by operations of the nature indicated above, the second inequality is said to follow *a fortiori* from the first.

§ 2. Formulas and rules for reckoning with ordinary complex numbers. Notations, definitions, and lemmas.

In the following, all numbers are ordinary complex numbers unless the contrary is expressly stated. As usual, u and $\bar{u}$ denote conjugate numbers, and $|u|$ the absolute value of u; we then have $u\bar{u} = |u|^2$. By $\Re(u) = \frac{1}{2}(u + \bar{u})$ we denote the real part of u, and by

$$\Im(u) = \frac{1}{2i}(u - \bar{u}) = \Re\left(\frac{u}{i}\right)$$

the coefficient of i in the purely imaginary part of u (read, for brevity, "imaginary part of u"), or

$$u = \Re(u) + i\Im(u).$$

Occasionally we also use the notation

$$\operatorname{sg}(u) = \frac{u}{|u|} \text{ (read "signum } u\text{")}$$

when $u \neq 0$, while $\operatorname{sg}(0) = 0$.

With these notations, there follows from $|u + v|^2 = (u + v)(\bar{u} + \bar{v})$ the identity

$$|u + v|^2 = |u|^2 + 2\Re(u\bar{v}) + |v|^2 *$$

which is of importance in the following. This identity leads to another

$$|\beta u + \alpha v|^2 - |\bar{\alpha}u + \bar{\beta}v|^2 = (|\beta|^2 - |\alpha|^2)(|u|^2 - |v|^2) †$$

* For easy reckoning with complex numbers the following identities are also useful

$$|u + v|^2 - |u - v|^2 = 4\Re(u\bar{v}) \quad \text{or} \quad |u|^2 - |v|^2 = \Re((u + v)(\bar{u} - \bar{v})),$$

$$|u + v|^2 - |u - \bar{v}|^2 = 4\Re(u)\Re(v) = |u + \bar{v}|^2 - |u - v|^2,$$

$$|u + v|^2 - |u + \bar{v}|^2 = 4\Im(u)\Im(v);$$

however, these identities will not be needed in the present paper.

† Other identities are

$$|\beta u + \alpha v|^2 + |\bar{\alpha}u - \bar{\beta}v|^2 = (|\alpha|^2 + |\beta|^2)(|u|^2 + |v|^2),$$

$$|\beta u + \alpha v|^2 - |\bar{\beta}u - \bar{\alpha}v|^2 = 4\Re(\alpha\bar{\beta})\Re(u\bar{v}),$$

$$|\beta u + \alpha v|^2 - |\bar{\beta}u + \bar{\alpha}v|^2 = 4\Im(\alpha\bar{\beta})\Im(u\bar{v}).$$

simply by observing that the terms in $\mathfrak{R}$ from the two squares to the left are identical. From the last identity, we shall now derive all the algebraic lemmas we need. It should be noted that since the right side of the identity contains only the absolute values of the four variables, we may change the arguments of α, β, u, v in any manner in the expression to the left without changing its value.

LEMMA 1. *When the complex number α is such that $|\alpha| < 1$, the following inequalities, in which inequality and equality signs read from top to bottom correspond to each other,*

$$|u| \lesseqqgtr |v| \qquad \text{and} \qquad |u + \alpha v| \lesseqqgtr |\bar{\alpha}u + v|$$

are equivalent.

When $|\alpha| > 1$, the same statement is true provided that one of the inequalities is reversed, i.e., the signs from top to bottom are read in the reversed order.

When finally $|\alpha| = 1$, we always have $|u + \alpha v| = |\bar{\alpha}u + v|$.

THE PROOF is read off at once from our identity by making $\beta = 1$ and comparing the signs of both sides.

NOTE. This lemma, which we shall also call *the α method,* is very useful for transforming inequalities between the absolute values of complex expressions. Throughout the following, we shall use the fact that it is frequently possible to determine α in such a manner that certain troublesome terms are eliminated from one side of an inequality, so that they will occur on the other side only.

LEMMA 2. *Assuming R to be real and positive,* our identity may be written thus:*

$$\begin{aligned} |R^2 - \bar{u}_0 u|^2 - |R(u - u_0)|^2 &= (R^2 - |u_0|^2)(R^2 - |u|^2) \\ &= (R^2 - |u_0 u|)^2 - R^2(|u| - |u_0|)^2 \\ &= (R^2 + |u_0 u|)^2 - R^2(|u| + |u_0|)^2. \end{aligned}$$

LEMMA 3. *From the preceding lemma, the following three cases are obtained by comparing signs in the upper line and dividing by $|R^2 - \bar{u}_0 u|^2$:*

(*a*) *For $|u_0| < R$, the inequalities*

$$|u| \lesseqqgtr R \qquad \text{and} \qquad \left|\frac{R(u - u_0)}{R^2 - \bar{u}_0 u}\right| \lesseqqgtr 1$$

are equivalent; for the lower inequality signs it is necessary however that $R^2 \neq \bar{u}_0 u$.

(*b*) *For $|u_0| > R$, the same statement is true provided that the signs in*

* As everywhere in the following.

the second inequality are reversed; for the upper inequality signs it is necessary however that $R^2 \neq \bar{u}_0 u$.

(c) *For* $|u_0| = R$, *we always have*

$$\left|\frac{R(u - u_0)}{R^2 - \bar{u}_0 u}\right| = 1;$$

for $|u| = R$ *it is necessary however that* $u \neq u_0$ (or what is the same in this case, $R^2 \neq \bar{u}_0 u$).

Lemma 4. *When* $R^2 \neq \bar{u}_0 u$ *and* k *is a real, non-negative number satisfying the inequality* $k|u_0| < R$, *the inequalities*

$$\left|\frac{R(u - u_0)}{R^2 - \bar{u}_0 u}\right| \lesseqqgtr k \qquad \text{and} \qquad \left|u - \frac{R^2(1 - k^2)}{R^2 - k^2|u_0|^2} u_0\right| \lesseqqgtr kR \left|\frac{R^2 - |u_0|^2}{R^2 - k^2|u_0|^2}\right|$$

are equivalent.

When $k|u_0| > R$, *this statement remains true when reversing the signs in the second inequality.*

Proof: The first inequality is equivalent to

$$|R(u - u_0)| \lesseqqgtr |k(R^2 - \bar{u}_0 u)|$$

and hence, by the α method, also to

$$|R(u - u_0) + \alpha k(R^2 - \bar{u}_0 u)| \lesseqqgtr |\bar{\alpha} R(u - u_0) + k(R^2 - \bar{u}_0 u)|$$

for $|\alpha| < 1$, while the signs are to be reversed for $|\alpha| > 1$. Making $\alpha = k|u_0|/R$, u is eliminated from the right side, and the lemma follows upon division by $|R^2 - k^2|u_0|^2| \cdot R^{-1}$, which is $\neq 0$ by our assumptions.

Note. For $k = 1$, lemma 3 (*a*) and (*b*) follows from this.

Lemma 5. *When* $|u_0| < R$ *and* $|u| < R$, *then*

$$\frac{R\,||u| - |u_0||}{R^2 - |u_0 u|} \leqq \left|\frac{R(u - u_0)}{R^2 - \bar{u}_0 u}\right| \leqq \frac{R(|u| + |u_0|)}{R^2 + |u_0 u|}.$$

The equality sign to the left will hold when and only when either one (or both) of the numbers u_0 *and* u *is zero, or* $\operatorname{sg}(u) = \operatorname{sg}(u_0)$, *and the equality sign to the right, when and only when either one (or both) of the numbers* u_0 *and* u *is zero or* $\operatorname{sg}(u) = -\operatorname{sg}(u_0)$.

Proof. By lemma 2, either side in the identities

$$|R^2 - \bar{u}_0 u|^2 - |R(u - u_0)|^2 = (R^2 - |u_0 u|)^2 - R^2(|u| - |u_0|)^2,$$

$$|R^2 - \bar{u}_0 u|^2 - |R(u - u_0)|^2 = (R^2 + |u_0 u|)^2 - R^2(|u| + |u_0|)^2$$

is positive. Dividing the upper and lower identity member by member by the inequalities

$$|R^2 - \bar{u}_0 u|^2 \geqq (R^2 - |u_0 u|)^2 \qquad \text{and} \qquad |R^2 - \bar{u}_0 u|^2 \leqq (R^2 + |u_0 u|)^2$$

respectively, inequalities are obtained which are equivalent to those to be proved. The statement regarding the equality signs follows from the fact that the equality signs in the inequalities used in the proof hold when and only when $\mathrm{sg}(\bar{u}_0 u) = 1$ or 0 and $\mathrm{sg}(\bar{u}_0 u) = -1$ or 0 respectively.

After these quite elementary algebraic lemmas* we now proceed to a function theoretic one; but first we shall introduce some definitions and notations which will be used everywhere in the following.

By x and x^* we denote complex variables satisfying the conditions $|x| < R$, $|x^*| < R$, unless the contrary is explicitly stated. Moreover, we denote by x_0 a particular value of x which may be chosen arbitrarily subject to the condition stated, viz. $|x_0| < R$. By $x_1, x_2, \cdots, x_n$ we denote other values *which will be defined in each individual case.*

With every x_ν $(\nu = 0, 1, 2, \cdots)$ we associate a real and non-negative κ_ν determined by

$$\kappa_\nu \equiv \kappa_\nu(x) = \left| \frac{R(x - x_\nu)}{R^2 - \bar{x}_\nu x} \right|.$$

It follows at once from lemma 3 (a) that $\kappa_0 < 1$ always, and that for other subscripts $\kappa_\nu \lesseqgtr 1$ according as $|x_\nu| \lesseqgtr R$; for the lower inequality sign it is necessary that $R^2 \neq \bar{x}_\nu x$. For brevity, we denote by $\kappa \equiv \kappa(x)$, a product of κ's with different subscripts, for instance $\kappa = \kappa_0\kappa_1 \cdots \kappa_n$. When all of $|x_1|, |x_2|, \cdots, |x_n|$ are $< R$, this κ will satisfy the inequality $0 \leqq \kappa < 1$. When using the other variable x^*, we introduce the similar notations $\kappa_\nu^* \equiv \kappa(x_\nu^*)$ for $\nu = 0, 1, 2, \cdots$ and $\kappa^* \equiv \kappa(x^*)$; the constants $x_0, x_1, x_2, \cdots$ we do not change unless expressly stated. A particularly simple case occurs when κ reduces to κ_0, which equals $|x|/R$ under the special assumption $x_0 = 0$. Lemma 3 (a) leads to an important result. When all $|x_\nu| < R$, the corresponding $\kappa_\nu = 1$ for $|x| = R$, and consequently $\kappa \to 1$ as $|x| \to R$.

By $y \equiv y(x)$ we always denote an analytic function of x which is holomorphic for $|x| < R$; for $|x| = R$ we make no assumptions. Evidently $y(x)$ may be expressed as a power series with a radius of convergence $\geqq R$. By y^* we denote $y(x^*)$, and for brevity, we always write $y_0 = y(x_0)$ except for $x_0 = 0$, when the unabridged notation $y(0)$ is used to avoid ambiguity.

Apart from the algebraic lemmas already set down, we use in this paper only a few propositions from the most elementary part of the theory of functions, which are proved most simply, elegantly and advantageously (i.e., with the smallest amount of presupposed theory) by means of power series. Such propositions are: the quotient of two holomorphic functions

* The reader will undoubtedly have noticed that lemmas 2–5 are homogeneous in form, so that it would have been sufficient to state them for $R = 1$. But this would bring no advantage whatever in the following, and the homogeneous form was chosen purposely.

is also a holomorphic function when the denominator vanishes nowhere for $|x| < R$; the maximum of the absolute value of a function holomorphic for $|x| \leqq r$ will occur only for an x-value (or values) such that $|x| = r$, unless the function is a constant, and so forth; and finally

Lemma 6, generalization of Schwarz' lemma. *When the function y is holomorphic for $|x| < R$ and satisfies the inequality $|y| < M$ for $|x| < R$, M being real and positive,* and when y has zeros at $x_1, x_2, \cdots, x_n$† in the region considered, then*

$$|y| \leqq M\kappa, \qquad \textit{where} \quad \kappa = \kappa_1\kappa_2\cdots\kappa_n.$$

This solution is complete (since this inequality leads a fortiori to $|y| < M$, and gives $y = 0$ for $\kappa = 0$ or $x = x_1, x_2, \cdots, x_n$).

Proof. It follows from our assumptions that the function

$$\frac{y}{(x - x_1)(x - x_2)\cdots(x - x_n)}$$

and therefore also

$$u = y\frac{(R^2 - \bar{x}_1x)(R^2 - \bar{x}_2x)\cdots(R^2 - \bar{x}_nx)}{R(x - x_1)\cdot R(x - x_2)\cdots R(x - x_n)}$$

are holomorphic for $|x| < R$, and we have

$$|u| = \frac{|y|}{\kappa} < \frac{M}{\kappa}.$$

It has already been noted that $\kappa \to 1$ as $|x| \to R$, and consequently we may choose an $|x| = r < R$ so close to R that $r > |x_\nu|$ for $\nu = 1, 2, \cdots, n$ and that $\kappa > 1/(1 + \epsilon)$, where ϵ is any assigned positive number; we then have $|u| < M(1 + \epsilon)$ on the circumference $|x| = r$. By a theorem in the elementary theory of functions quoted above, the last inequality then also holds for all $|x| \leqq r$, and since ϵ is arbitrary, it is seen that for $|x| \leqq r < R$ we have $|u| \leqq M$ or $|y| \leqq M\kappa$, which completes the proof.

Note.‡ The equality sign will obviously occur when $x = x_1$ or $x_2 \cdots$ or x_n, both sides being then equal to zero. Suppose however that $|y| = M\kappa$ for *one* value $x = x_0$ distinct from $x_1, x_2, \cdots, x_n$ and such that $|x_0| < R$. Then $|u| = |y|/\kappa = M$ for $x = x_0$; on the other hand, we have proved that $|u| \leqq M$ for $|x| < R$. Since $|x_0| < \frac{1}{2}(R + |x_0|) < R$, u is holomorphic inside and on the circumference $|x| = \frac{1}{2}(R + |x_0|)$, and $|u|$ takes its maximum value M at the point x_0 inside this circumference;

* As everywhere in the following.

† No x_ν may occur more times than indicated by the multiplicity of the corresponding zero of y. This condition is always understood in the following without being expressly stated in each case.

‡ In the translation, this note has been changed so as to include the proof that the special rational function given below is the only one for which $|y| = M\kappa$.

by the function theoretic theorem quoted before, u is therefore a constant, and since $|u(x_0)| = M$, we have $u = \gamma M$, where γ is a constant such that $|\gamma| = 1$. Returning to the function y, it follows that

$$y = \gamma M \prod_{\nu=1}^{n} \frac{R(x - x_\nu)}{R^2 - \bar{x}_\nu x}.$$

Conversely, it is seen at once that this y is holomorphic for $|x| < R$, and that for *every* x where $|x| < R$, we have $|y| = M\kappa$ (and hence $|y| < M$). Since $\kappa \to 1$ as $|x| \to R$, we also have $|y| \to M$.

Before proceeding to apply the lemmas proved in this paragraph, we shall consider the last of them for a moment, writing it out at length:

When $y(x)$ is holomorphic and $|y| < M$ for $|x| < R$, and when $x_1, x_2, \cdots, x_n$ are some of the zeros of the function in the region considered, then

$$M \geqq \left| y(x) \frac{(R^2 - \bar{x}_1 x)(R^2 - \bar{x}_2 x)\cdots(R^2 - \bar{x}_n x)}{R(x - x_1)\cdot R(x - x_2)\cdots R(x - x_n)} \right|.$$

When moreover y is holomorphic in a larger region $|x| < R'(R' > R)$, the lemmas 3 (c), 3 (b) and 3 (a) allow us to include a fortiori among the x_ν occurring to the right any number of other x_ν situated quite arbitrarily in the extended region outside the original one, the corresponding factors $1/\kappa_\nu$ being $\leqq 1$ for $|x| < R$ (and since $|x| < R$, it is unnecessary to add the restriction $x \neq x_\nu$ for these new x_ν). Of course we are at liberty to let these x_ν coincide with any existing zeros of $y(x)$ in the new region.

This extension of lemma 6 includes, for $x = 0$, my theorem quoted above in § 1 no. 3, as well as the results of Carathéodory and Fejér in the paper quoted in [8].

Lemma 6 may also be extended in another direction. It is no essentially new problem to consider zeros of $y - \eta$ instead of zeros of y, η being an arbitrary constant subject however to the condition $|\eta| < M$. All that is necessary is to replace y by the function

$$\frac{M(y - \eta)}{M^2 - \bar{\eta} y}.$$

In fact, this function is holomorphic, our assumptions showing that the absolute value of the denominator is greater than zero, and by lemma 3 (a), the absolute value of the function is less than unity. We may therefore apply lemma 6, and thus find the following extension of the latter:

When y is a holomorphic function of x for $|x| < R$, and $|y| < M$ for $|x| < R$, and $x_1, x_2, \cdots, x_n$ are some of the zeros (interior to this region) of the function $y - \eta$, where η is a constant such that $|\eta| < M$, then*

$$M\,|y - \eta| \leqq |M^2 - \bar{\eta} y|\,\kappa,$$

where $\kappa = \kappa_1\kappa_2\cdots\kappa_n$.

* When $|\eta| \geqq M$, the region considered contains no zeros of $y - \eta$.

For $\eta = 0$, we again have lemma 6. What is said above about extending the latter will of course apply, mutatis mutandis, here also.

From $\kappa = 0$ we conclude that $y(x_\nu) = \eta$ for $\nu = 1, 2, \cdots, n$, and from the theorem it follows a fortiori that $M \mid y - \eta \mid < \mid M^2 - \bar{\eta} y \mid$ or $\mid y \mid < M$ (by lemma 3 (*a*)). Hence the theorem is complete.

For $x = 0$ we obtain the result of Landau [9], p. 107; see above, p. 3.

§ 3. General theorems concerning functions bounded with respect to their absolute value.

In addition to the notations and definitions introduced in § 2 (p. 9) we shall assume throughout this paragraph that $\mid y \mid < M$ for $\mid x \mid < R$, unless otherwise stated.

THEOREM 1. *When x_0 is fixed arbitrarily* within the region, and $x_0, x_1, \cdots, x_n$ denote some of the zeros of $y - y_0$ in the region (in particular, none of them except x_0), then we have the following three equivalent inequalities*

I. $$M \mid y - y_0 \mid \leqq \mid M^2 - \bar{y}_0 y \mid \kappa;$$

II. $$\mid y - y_0 \mid^2 \leqq \frac{(M^2 - \mid y_0 \mid^2)(M^2 - \mid y \mid^2)}{M^2} \frac{\kappa^2}{1 - \kappa^2};$$

III. $$\left| y - y_0 \frac{M^2(1 - \kappa^2)}{M^2 - \mid y_0 \mid^2 \kappa^2} \right| \leqq M \frac{M^2 - \mid y_0 \mid^2}{M^2 - \mid y_0 \mid^2 \kappa^2} \kappa;$$

where

$$\kappa = \kappa_0 \kappa_1 \cdots \kappa_n.$$

The theorem is complete, i.e., all the assumptions follow again from either one of the inequalities I, II *or* III, *provided that there exists an x_0 for which $\mid y_0 \mid < M$.*

PROOF. On account of the assumptions, the function $y - y_0$ has zeros for $x = x_0, x_1, \cdots, x_n$, and I is therefore an immediate consequence of lemma 6 in the extended form given at the end of § 2. Supposing conversely that inequality I is true, then $y = y_0$ for $\kappa = 0$, and from I it follows a fortiori that $M \mid y - y_0 \mid < \mid M^2 - \bar{y}_0 y \mid$. When there exists an x_0 for which $\mid y_0 \mid < M$, then we also have $\mid y \mid < M$ by lemma 3 (*a*), and consequently I is complete.

Squaring both sides of I and using the identity in lemma 2, it follows that equivalently

$$M^2 \mid y - y_0 \mid^2 \leqq \mid M^2 - \bar{y}_0 y \mid^2 \kappa^2$$
$$= [M^2 \mid y - y_0 \mid^2 + (M^2 - \mid y_0 \mid^2)(M^2 - \mid y \mid^2)]\kappa^2,$$

and this is equivalent to II.

* In the applications it is of course most advantageous to choose x_0 so that $y_0 \equiv y(x_0)$ has a convenient value.

Applying lemma 4 to I (writing y for u, y_0 for u_0, M for R and κ for k) we find, since $|y_0|\kappa < |y_0| < M$, that I and III are equivalent. Thus theorem 1 is proved in its entirety.

NOTE 1. I and II are also true under the assumptions $|y| > M$, $|y_0| > M$, as is seen at once upon replacing y by $1/y$ and y_0 by $1/y_0$; this also follows from lemma 3 (b).

NOTE 2.* The equality sign always holds in I, II and III for the particular values $x = x_0$ or $x_1 \cdots$ or x_n. If it holds for an x distinct from these, it follows from the note to lemma 6 that y is of the form

$$y = M \frac{y_0 + \gamma M R^{n+1} \prod_{\nu=0}^{n} \frac{x - x_\nu}{R^2 - \bar{x}_\nu x}}{M + \gamma \bar{y}_0 R^{n+1} \prod_{\nu=0}^{n} \frac{x - x_\nu}{R^2 - \bar{x}_\nu x}},$$

where $|y_0| < M$ and $|\gamma| = 1$, and conversely, the equality sign holds in I, II and III for all x such that $|x| < R$ when y is of the form indicated.

NOTE 3. For $y_0 = 0$, I, II and III reduce to lemma 6.

THEOREM 2. *Under the same assumptions as in theorem I we have*

$$(a) \qquad |y - y_0| \leqq \frac{M^2 - |y_0|^2}{M - |y_0|\kappa}\kappa;$$

$$(b) \qquad |y - y_0| \leqq (M + |y_0|\kappa)\kappa;$$

$$(c) \qquad |y - y_0| \leqq \sqrt{M^2 - |y_0|^2}\frac{\kappa}{\sqrt{1 - \kappa^2}}.$$

PROOF. (a) follows from theorem 1, I upon replacing

$$|M^2 - \bar{y}_0 y| = |M^2 - |y_0|^2 - \bar{y}_0(y - y_0)|$$

a fortiori by $M^2 - |y_0|^2 + |y_0||y - y_0|$. (b) follows a fortiori from (a), and (c) a fortiori from theorem 1, II.

NOTE 1. When an x_0 exists for which $y_0 = 0$, (a) and (b) are evidently complete.

NOTE 2. When

$$\kappa = \kappa_0 = \frac{R|x - x_0|}{R^2 - \bar{x}_0 x},$$

it is easy to derive from (a), (b) or (c) upper bounds for the absolute value of the difference quotient $(y - y_0)/(x - x_0)$. For $x \to x_0$, the first of these gives Lindelöf's formula (upon replacing x_0 by x)

$$\left|\frac{dy}{dx}\right| \leqq \frac{R(M^2 - |y|^2)}{M(R^2 - |x|^2)}.\dagger$$

* This note has been changed to correspond to the changes in the note to lemma 6.

† See above, p. 4.

We shall return to this in the following in more general (and better) forms.

THEOREM 3. *Under the same assumptions as in theorem* 1 *we have, η being a constant,*

$$\left| \eta - y_0 \frac{M^2(1-\kappa^2)}{M^2 - |y_0|^2\kappa^2} \right| - M\frac{M^2 - |y_0|^2}{M^2 - |y_0|^2\kappa^2}\kappa \leqq |y - \eta|$$

$$\leqq \left| \eta - y_0 \frac{M^2(1-\kappa^2)}{M^2 - |y_0|^2\kappa^2} \right| + M\frac{M^2 - |y_0|^2}{M^2 - |y_0|^2\kappa^2}\kappa.$$

THE PROOF of the two inequalities follows from theorem 1, III upon replacing the left side by

$$\left| \eta - y_0 \frac{M^2(1-\kappa^2)}{M^2 - |y_0|^2\kappa^2} \right| - |y - \eta|$$

and

$$|y - \eta| - \left| \eta - y_0 \frac{M^2(1-\kappa^2)}{M^2 - |y_0|^2\kappa^2} \right|$$

respectively.

NOTE 1. For $\kappa = 0$, both inequalities reduce to the identity

$$|y(x_\nu) - \eta| = |y_0 - \eta|.$$

NOTE 2. For $\eta = y_0$, the inequality to the right reduces to theorem 2, (a).

COROLLARY 1. When $\eta = 0$, theorem 3 takes the following form (after a simple reduction to the right and left)

$$M\frac{|y_0| - M\kappa}{M - |y_0|\kappa} \leqq |y| \leqq M\frac{|y_0| + M\kappa}{M + |y_0|\kappa}.$$

(Before going further, we shall prove this corollary in a different manner by the inequality to the left in lemma 5. From our assumptions, it follows that

$$\left| \frac{M(y - y_0)}{M^2 - \bar{y}_0 y} \right| \leqq \kappa,$$

or a fortiori

$$\frac{M(|y_0| - |y|)}{M^2 - |y_0||y|} \leqq \kappa \quad \text{and} \quad \frac{M(|y| - |y_0|)}{M^2 - |y_0||y|} \leqq \kappa,$$

and from these, the inequalities to be proved follow equivalently by solving in respect to $|y|$.)

Moreover, it may be noted that when there exists an x_0 such that $y_0 = 0$, the inequality to the right reduces (for this value of x_0) to lemma 6, the generalized lemma of Schwarz, and is then complete. Besides, we

always have

$$M\frac{|y_0|+M\kappa}{M+|y_0|\kappa}=M\left(1-\frac{(M-|y_0|)(1-\kappa)}{M+|y_0|\kappa}\right)<M$$

when

$$|y_0|<M.$$

A noteworthy special case is $\kappa=\kappa_0$,* or

$$M\frac{|y_0||R^2-\bar{x}_0x|-MR|x-x_0|}{M|R^2-\bar{x}_0x|-R|y_0(x-x_0)|}\leqq|y|$$

$$\leqq M\frac{|y_0||R^2-\bar{x}_0x|+MR|x-x_0|}{M|R^2-\bar{x}_0x|+R|y_0(x-x_0)|},$$

which reduces, for $x_0=0$ (and $y_0=y(0)$), to two of Lindelöf's formulas (l.c. [11]), formula (4) p. 12,† and formula (1), p. 11, compare above, p. 4).

From corollary 1 we deduce

COROLLARY 2.

$$M^2\frac{M^2-|y_0|^2}{M^2-|y_0|^2\kappa^2}(1-\kappa)^2\leqq M^2-|y|^2\leqq M^2\frac{M^2-|y_0|^2}{M^2-|y_0|^2\kappa^2}(1+\kappa)^2.$$

PROOF. Corollary 1 may be written

$$M\frac{M+|y_0|}{M-|y_0|\kappa}(1-\kappa)\leqq M+|y|\leqq M\frac{M+|y_0|}{M+|y_0|\kappa}(1+\kappa)$$

and

$$M\frac{M-|y_0|}{M+|y_0|\kappa}(1-\kappa)\leqq M-|y|\leqq M\frac{M-|y_0|}{M-|y_0|\kappa}(1+\kappa),$$

whence corollary 2 follows upon multiplication member by member. In the following, we shall have occasion to use the inequality to the right in this corollary; but it should be stated explicitly that this cannot be used to advantage unless κ has such a value that the right-hand member is $<M^2$; otherwise the inequality is trivial, and it is better simply to use M^2. Since we may replace the right-hand member a fortiori by

$$M^2\frac{M^2-|y_0|^2}{M^2(1-\kappa^2)}(1+\kappa)^2=(M^2-|y_0|^2)\frac{1+\kappa}{1-\kappa},$$

it is seen that the formula is advantageous when

$$\kappa<\frac{|y_0|^2}{2M^2-|y_0|^2},$$

or at any rate for

$$\kappa<\tfrac{1}{2}\frac{|y_0|^2}{M^2}.$$

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* Where consequently no assumption is made regarding the existence of the zeros $x_1, x_2, \cdots$.

† From which however a superfluous restriction is removed.

COROLLARY 3. When $0 < M' < | y | < M$, then

$$M' \frac{| y_0 | + M'\kappa}{M' + | y_0 | \kappa} \leqq | y | \leqq M \frac{| y_0 | + M\kappa}{M + | y_0 | \kappa}.$$

PROOF. The inequality to the right has already been proved in corollary 1, and the one to the left follows by applying the inequality to the right to the function $1/y$ and replacing y_0 and M by $1/y_0$ and $1/M'$.

THEOREM 4. *Under the same assumptions as in theorem* 1, *and making* η $(\neq y_0)$ *a constant less than* M *in absolute value,* then*

$$y \neq \eta \qquad \textit{for} \qquad \kappa = \kappa_0\kappa_1 \cdots \kappa_n < \left| \frac{M(y_0 - \eta)}{M^2 - \bar{\eta}y_0} \right|.\dagger$$

PROOF. $y \neq \eta$ when the first member in theorem 3 is positive, or when

$$\left| \eta - y_0 \frac{M^2(1 - \kappa^2)}{M^2 - | y_0 |^2\kappa^2} \right| > M \frac{M^2 - | y_0 |^2}{M^2 - | y_0 |^2\kappa^2} \kappa.$$

By lemma 4 (writing κ for k, M for R, η for u and y_0 for u_0; it follows from our assumptions that $M^2 \neq \bar{y}_0\eta$ and $\kappa | y_0 | < M$) this inequality is equivalent to

$$\left| \frac{M(\eta - y_0)}{M^2 - \bar{y}_0\eta} \right| = \left| \frac{M(y_0 - \eta)}{M^2 - \bar{\eta}y_0} \right| > \kappa,$$

which proves the theorem.

Another proof is obtained by applying the left side of corollary 1, theorem 3, to the function

$$\frac{M(y - \eta)}{M^2 - \bar{\eta}y},$$

which is less than unity in absolute value (as shown at the end of § 2), and which moreover takes the value

$$\frac{M(y_0 - \eta)}{M^2 - \bar{\eta}y_0}$$

for $x = x_\nu$ $(\nu = 0, 1, 2, \cdots, n)$, whence

$$\left| \frac{M(y - \eta)}{M^2 - \bar{\eta}y} \right| \geqq \frac{\left| \frac{M(y_0 - \eta)}{M^2 - \bar{\eta}y_0} \right| - \kappa}{1 - \left| \frac{M(y_0 - \eta)}{M^2 - \bar{\eta}y_0} \right| \kappa}.$$

* For $| \eta | \geqq M$, we have $y \neq \eta$ in the entire region $| x | < R$.

† By lemma 2, this condition may also be written

$$\kappa^2 < \frac{M^2 | y_0 - \eta |^2}{M^2 | y_0 - \eta |^2 + (M^2 - | \eta |^2)(M^2 - | y_0 |^2)}.$$

NOTE. It is apparent at once that the more factors it is possible to put into κ on the left in the inequality in theorem 4, the larger will be the region it determines for x. The significance of the theorem is thus greatest when we know as many of the zeros of $y - y_0$ as possible; but we may always make $\kappa = \kappa_0$ and obtain

COROLLARY 1.

$$y \neq \eta \qquad \text{for} \qquad \kappa_0 = \left|\frac{R(x - x_0)}{R^2 - \bar{x}_0 x}\right| < \left|\frac{M(y_0 - \eta)}{M^2 - \bar{\eta} y_0}\right|.$$

We note the particular case $x_0 = 0$ (and $y_0 = y(0)$), which gives the result due to Landau (l.c.[9], p. 105) quoted p. 3 above.

Another noteworthy special case is $\eta = 0$, which gives

$$y \neq 0 \qquad \text{for} \qquad \left|\frac{R(x - x_0)}{R^2 - \bar{x}_0 x}\right| < \frac{|y_0|}{M}.$$

Hence we obtain a fortiori, replacing

$$|R^2 - \bar{x}_0 x| = |R^2 - |x_0|^2 - \bar{x}_0(x - x_0)|$$

by $R^2 - |x_0|^2 - |x_0|\,|x - x_0|$, the result of Lindelöf (l.c.[11], formula (5), p. 12):

$$y \neq 0 \qquad \text{for} \qquad |x - x_0| < \frac{R^2 - |x_0|^2}{RM + |x_0 y_0|}\,|y_0|.$$

COROLLARY 2. The preceding corollary may be transformed equivalently into

$$y \neq \eta \qquad \text{for} \qquad \left|x - x_0 \frac{R^2(|M^2 - \bar{\eta} y_0|^2 - M^2|y_0 - \eta|^2)}{R^2|M^2 - \bar{\eta} y_0|^2 - M^2|x_0(y_0 - \eta)|^2}\right|$$

$$< \frac{RM(R^2 - |x_0|^2)\,|M^2 - \bar{\eta} y_0|\,|y_0 - \eta|}{R^2|M^2 - \bar{\eta} y_0|^2 - M^2|x_0(y_0 - \eta)|^2},$$

and in particular

$$y \neq 0 \qquad \text{for} \qquad \left|x - x_0 \frac{R^2(M^2 - |y_0|^2)}{R^2 M^2 - |x_0 y_0|^2}\right| < \frac{RM(R^2 - |x_0|^2)\,|y_0|}{R^2 M^2 - |x_0 y_0|^2}.$$

The last region is larger in every direction than the one due to Lindelöf which we have just quoted.

THE PROOF follows by using lemma 4 for the transformation of corollary 1 (writing $|[M(y_0 - \eta)]/[M^2 - \bar{\eta} y_0]|$ for k, x for u, and x_0 for u_0; we then have

$$|x_0| \left|\frac{M(y_0 - \eta)}{M^2 - \bar{\eta} y_0}\right| < |x_0| < R).$$

Using the identity in lemma 2, we also have

$$y \neq \eta \text{ for } \left| x - x_0 \frac{R^2(M^2 - |\eta|^2)(M^2 - |y_0|^2)}{R^2(M^2 - |\eta|^2)(M^2 - |y_0|^2) + M^2|y_0 - \eta|^2(R^2 - |x_0|^2)} \right|$$
$$< \frac{RM(R^2 - |x_0|^2)\,|M^2 - \bar{\eta}y_0|\,|y_0 - \eta|}{R^2(M^2 - |\eta|^2)(M^2 - |y_0|^2) + M^2|y_0 - \eta|^2(R^2 - |x_0|^2)}.$$

Theorem 5, concerning the derivative of y. *Under the same assumptions as in theorem* 1, *we have*

$$\left|\frac{dy}{dx}\right| \leqq \frac{R(M^2 - |y|^2)}{M(R^2 - |x|^2)} \leqq \frac{RM}{R^2 - |x|^2}\frac{M^2 - |y_0|^2}{M^2 - |y_0|^2\kappa^2}(1 + \kappa)^2 *$$

Proof. As we have noted before, the first inequality, Lindelöf's formula, is an immediate consequence of theorem 1, I or II, making $\kappa = \kappa_0$ and $x_0 \to x$. The second inequality follows from theorem 3, corollary 2, inequality to the right.

Note 1. The inequality obtained a fortiori by leaving out the middle member may be said to be more extensive in a certain sense than Lindelöf's formula, since the latter results from the former for $x_0 = x$. Moreover, the third member contains not y but y_0, which may be regarded as known.

Note 2.† To find when the equality sign holds in this theorem, let us assume that for $x = x_0$ and $y = y_0$ ($|x_0| < R$, $|y_0| < M$) we have

$$\left|\frac{dy}{dx}\right|_0 = \frac{R(M^2 - |y_0|^2)}{M(R^2 - |x_0|^2)}.$$

The function

$$u = \frac{M(y - y_0)}{M^2 - \bar{y}_0 y}\frac{R^2 - \bar{x}_0 x}{R(x - x_0)}$$

is holomorphic for $|x| < R$, and making $\kappa = \kappa_0$ in theorem 1, I, we find $|u| \leqq 1$ for $|x| < R$. Moreover,

$$u(x_0) = \lim_{x \to x_0} u = \frac{M(R^2 - |x_0|^2)}{R(M^2 - |y_0|^2)}\lim_{x \to x_0}\frac{y - y_0}{x - x_0} = \frac{M(R^2 - |x_0|^2)}{R(M^2 - |y_0|^2)}\left(\frac{dy}{dx}\right)_0,$$

and using the value of $|dy/dx|_0$, it is seen that $|u(x_0)| = 1$. Since $|x_0| < \frac{1}{2}(R + |x_0|) < R$, u is holomorphic inside and on the circumference

* Instead of the first inequality we may write a fortiori

$$\left|\frac{dy}{dx}\right| \leqq \frac{RM}{R^2 - |x|^2}.$$

When it is also known that $|y| > M' > 0$ in the region considered, we have the stronger inequality

$$\left|\frac{dy}{dx}\right| < \frac{R(M^2 - M'^2)}{M(R^2 - |x|^2)}.$$

† This note has been changed in the translation to correspond to the changes in the note to lemma 6.

$|x| = \frac{1}{2}(R + |x_0|)$, and $|u|$ takes its maximum value unity at the point x_0 inside this circumference; by the function theoretic theorem quoted immediately before lemma 6, u is therefore a constant $= \gamma$, where $|\gamma| = 1$. Substituting $u = \gamma$ in the equation defining u and solving for y, we obtain

$$y = M\frac{y_0(R^2 - \bar{x}_0 x) + \gamma MR(x - x_0)}{M(R^2 - \bar{x}_0 x) + \gamma\bar{y}_0 R(x - x_0)}.$$

Retracing the steps of the proof, it is seen conversely that for any y of this form, where $|x_0| < R$, $|y_0| < M$, $|\gamma| = 1$, the equality sign holds to the left in our theorem when $x = x_0$.

NOTE 3. We have observed before (see above, p. 4) that Lindelöf's formula contains for $x = 0$ an earlier formula due to Landau which, writing y as a power series in x

$$y(x) = c_0 + c_1 x + c_2 x^2 + \cdots$$

with radius of convergence $\geqq R$, may be expressed thus:

$$RM\,|c_1| \leqq M^2 - |c_0|^2.^*$$

It is easy to give a more general form to this result. Let p and q be integers, $q > p \geqq 0$ and let θ be a primitive qth root of unity; then

$$\frac{1}{q}\sum_{\nu=1}^{q}\theta^{-p\nu}y(x\theta^{\nu}) = x^p(c_p + c_{p+q}x^q + c_{p+2q}x^{2q} + \cdots)$$

where the expression to the left is less than M in absolute value, and the series in parentheses to the right, considered as a power series in x^q, converges at least for $|x^q| < R^q$. Considering all values of x for which $|x| = r < R$, the series in parentheses will be $< M/r^p$ in absolute value for all these x, and therefore also for all $|x| < r$. But r may be taken as close to R as we please, and the absolute value of the series will therefore be $\leqq M/R^p$. Hence, by Landau's theorem

$$R^q\frac{M}{R^p}|c_{p+q}| \leqq \frac{M^2}{R^{2p}} - |c_p|^2,$$

or

$$R^{p+q}M\,|c_{p+q}| \leqq M^2 - R^{2p}|c_p|^2,$$

* That Landau (l. c.[2]) proves this formula for $R = 1$, $M = 1$, implies no restriction. Incidentally, Lindelöf's more general formula may be derived from Landau's by a linear substitution. Writing

$$y(x) = f\left(\frac{R^2(x + x_0)}{R^2 + \bar{x}_0 x}\right),$$

then, by lemma 3a, $|f(x)| < M$ for $|x| < R$, and $f(x)$ is seen to be holomorphic in this region. The formula $RM\,|y'(0)| \leqq M^2 - |y(0)|^2$ becomes

$$\frac{M}{R}|f'(x_0)|(R^2 - |x_0|^2) \leqq M^2 - |f(x_0)|^2,$$

which gives Lindelöf's formula upon replacing $f(x)$ by $y(x)$ and x_0 by x.

which is the relation we desired to prove.* In the particular case $p = 0$, we obtain the generalization of Landau's theorem due to Wiener and quoted on p. 4.

Theorem 6, concerning the difference quotient of y. *Under the same assumptions as in theorem* 1, *we have for* $x \neq x^*$

$$\left|\frac{y - y^*}{x - x^*}\right|^2 \leqq \frac{R^2(M^2 - |y|^2)(M^2 - |y^*|^2)}{M^2(R^2 - |x|^2)(R^2 - |x^*|^2)} \dagger$$

where x^* is a new independent variable, and $y^* \equiv y(x^*)$. When, conversely, this inequality is satisfied, and there exists a certain value of x for which $|y| < M$, then $|y| < M$ for all x.

Proof. In theorem 1, II, we make

$$\kappa = \kappa_0 = \frac{R|x - x_0|}{|R^2 - \bar{x}_0 x|},$$

replace x_0 by x^* and obtain

$$\left|\frac{y - y^*}{x - x^*}\right|^2 \leqq \frac{R^2(M^2 - |y|^2)(M^2 - |y^*|^2)}{M^2(|R^2 - \bar{x}^* x|^2 - R^2|x - x^*|^2)};$$

applying the identity in lemma 2, this becomes the desired inequality. When there exists a certain value of x^* for which $|y^*| < M$, we find conversely by theorem 1 that the inequality of theorem 6 is equivalent to

$$\left|\frac{M(y - y^*)}{M^2 - \bar{y}^* y}\right| \leqq \left|\frac{R(x - x^*)}{R^2 - \bar{x}^* x}\right|,$$

whence a fortiori

$$\left|\frac{M(y - y^*)}{M^2 - \bar{y}^* y}\right| < 1,$$

and this is equivalent to $|y| < M$ by lemma 3 (a). Thus the proof of theorem 6 is complete.

Note 1. By note 1 to theorem 1 it is readily seen that when there exists a certain value of x^* for which $|y^*| > M$, then the inequality in theorem 6 will also imply that $|y| > M$ for all values of x in the region considered.

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* It is easy to generalize this result considerably. But this would fall beyond the scope of the present paper, and must be reserved for another occasion.

† Hence a fortiori the simple formula

$$\left|\frac{y - y^*}{x - x^*}\right| \leqq \frac{RM}{\sqrt{(R^2 - |x|^2)(R^2 - |x^*|^2)}}.$$

When it is also known that $|y| > M' > 0$ in the region considered, we have the more accurate formula

$$\left|\frac{y - y^*}{x - x^*}\right| < \frac{R(M^2 - M'^2)}{M\sqrt{(R^2 - |x|^2)(R^2 - |x^*|^2)}}.$$

NOTE 2. When it is desirable to avoid y or y^* to the right in the inequality in theorem 6, we may eliminate either or both of these functions as before, so that there occurs instead a y_0 which may be preferable. To this purpose, we replace a fortiori

$$M^2 - |y|^2 \quad \text{by} \quad M^2 \frac{M^2 - |y_0|^2}{M^2 - |y_0|^2\kappa^2}(1+\kappa)^2,$$

(theorem 3, corollary 2), and

$$M^2 - |y^*|^2 \quad \text{by} \quad M^2 \frac{M^2 - |y_0|^2}{M^2 - |y_0|^2\kappa^{*2}}(1+\kappa^*)^2,$$

where

$$\kappa^* \equiv \kappa(x^*).$$

Evidently this may be varied in many ways, for instance by replacing κ by κ_0 and κ^* by the κ_0^* obtained by changing x_0 into x_0^*, which necessitates replacing y_0 by $y(x_0^*)$ in the second of the expressions above. The formula thus obtained—without assumptions regarding $x_1, x_2, \cdots$—is very general. For $x_0 = x$ and $x_0^* = x^*$ this of course reduces to theorem 6.

The above will undoubtedly be sufficient to show how we may proceed in other cases when an upper or lower bound of $|y|$ is given.

A last general remark may be appropriate. Let us assume that under the assumptions in theorem 1 (viz., $|y| < M$ for $|x| < R$, $|x_0| < R$ and $x_1, x_2, \cdots, x_n$ some of the zeros of $y - y_0$ in the given region) we have found an expression (non-analytic) in $y, y_0, M, x, x_0, x_1, \cdots, x_n, R$ satisfying the condition

$$\text{(I)} \qquad U\left(\frac{y}{M}, \frac{y_0}{M}\right) \geqq 0.$$

Since, for η constant and $|\eta| < M$, $|y| < M$ is equivalent to

$$\left|\frac{M(y-\eta)}{M^2 - \bar{\eta}y}\right| < 1,$$

and $M(y-\eta)/(M^2 - \bar{\eta}y)$ takes the value $M(y_0 - \eta)/(M^2 - \bar{\eta}y_0)$ for $x = x_0, x_1, \cdots, x_n$, the more general formula

$$\text{(II)} \qquad U\left(\frac{M(y-\eta)}{M^2 - \bar{\eta}y}, \frac{M(y_0-\eta)}{M^2 - \bar{\eta}y_0}\right) \geqq 0$$

follows from (I), which in turn results from (II) by making $\eta = 0$. Again, making $\eta = y_0$, we find the simplified formula

$$U\left(\frac{M(y-y_0)}{M^2 - \bar{y}_0 y}, 0\right) \geqq 0.$$

This may be varied in many ways. As we have already done in particular cases, η may be taken as a function of another independent variable x,* particularly when $x_1, x_2, \cdots, x_n$ are disregarded.

Although the theorems in the next paragraph concerning a bounded $\Re(y)$ become in fact more or less direct corollaries to the developments of the present paragraph, when using the formulas for the connection between absolute value, real part, etc., given at the beginning of § 2, the details are of sufficient interest to justify us in dealing at some length with theorems of this kind.

§ 4. General theorems concerning functions bounded with respect to their real part.

In addition to the notations and definitions introduced in § 2, we shall assume throughout this paragraph that, unless otherwise stated, $\Re(y) < A$ for $|x| < R$, where A denotes a real number.

THEOREM 1. *When x_0 is fixed arbitrarily within the region $|x| < R$, and $x_0, x_1, \cdots, x_n$ denote some of the zeros of the function $y - y_0$ in this region (in particular, none of them except x_0), then we have the following three equivalent inequalities*:

I. $$|y - y_0| \leqq |y - y_0 - 2(A - \Re y_0)|\,\kappa;$$

II. $$|y - y_0|^2 \leqq 4(A - \Re y_0)(A - \Re y)\frac{\kappa^2}{1 - \kappa^2};^*$$

III. $$\left|y - y_0 + 2(A - \Re y_0)\frac{\kappa^2}{1 - \kappa^2}\right| \leqq 2(A - \Re y_0)\frac{\kappa}{1 - \kappa^2},$$

where $\kappa = \kappa_0\kappa_1\cdots\kappa_n$.

The theorem is complete, i.e., all the assumptions follow again from either one of the inequalities I, II *or* III, *provided that there exists an x_0 for which $\Re(y_0) < A$.*

PROOF. We write for brevity (as in all the following proofs)

$$a = A - \Re(y), \qquad a_0 = A - \Re(y_0).$$

Then $a > 0$, $a_0 > 0$ and $\Re(y - y_0) = a_0 - a < a_0$, which is equivalent to $4a_0\Re(y - y_0) < 4a_0^2$ and to

$$|y - y_0|^2 < |y - y_0|^2 - 4a_0\Re(y - y_0) + 4a_0^2 = |y - y_0 - 2a_0|^2$$

by the first identity in § 2. Consequently, since $\Re(y - y_0 - 2a_0) < -a_0 < 0$,

$$\frac{y - y_0}{y - y_0 - 2a_0}$$

is a function of x holomorphic in the given region; moreover it is < 1 in

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* Or

$$\kappa^2 \geqq \frac{|y - y_0|^2}{|y - y_0|^2 + 4(A - \Re y_0)(A - \Re y)}.$$

absolute value and has the zeros x_0 and $x_1, x_2, \cdots, x_n$, if the latter are assumed to exist. Lemma 6 therefore gives

$$\left|\frac{y - y_0}{y - y_0 - 2a_0}\right| \leqq \kappa, \qquad \kappa = \kappa_0\kappa_1 \cdots \kappa_n,$$

which is equivalent to I. Conversely, when we assume that inequality I is true and that there exists an x_0 for which $\Re(y_0) < A$, we have $y = y_0$ for $\kappa = 0$, and also, a fortiori, $| y - y_0 | < | y - y_0 - 2a_0 |$ which we have just shown to be equivalent to $\Re(y - y_0) < a_0$ or $\Re(y) < A$; the inequality I is therefore complete.

Squaring I, we have the equivalent inequality

$$| y - y_0 |^2 \leqq (| y - y_0 |^2 - 4a_0\Re(y - y_0) + 4a_0^2)\kappa^2 = | y - y_0 |^2\kappa^2 + 4a_0a\kappa^2$$

which is equivalent to II.

III may also be proved by squaring, but we prefer the α method. For $| \alpha | < 1$, inequality I is equivalent to

$$| y - y_0 + \alpha(y - y_0 - 2a_0)\kappa | \leqq | \bar{\alpha}(y - y_0) + (y - y_0 - 2a_0)\kappa |$$

which, for $\alpha = \bar{\alpha} = -\kappa$ (which satisfies the condition $| \alpha | < 1$), is equivalent to

$$| (y - y_0)(1 - \kappa^2) + 2a_0\kappa^2 | \leqq 2a_0\kappa,$$

which again is equivalent to III. Thus theorem I is completely proved.

Note 1. When we replace y by $-y$ and A by $-A$ in theorem 1, it is seen that I and II remain unchanged; these two inequalities are therefore also true under the assumption $\Re(y) > A$ for $| x | < R$.

Note 2.* The equality sign in I, II and III always holds for $\kappa = 0$ or $x = x_0, x_1, \cdots, x_n$. If it holds for an x distinct from these, it follows from the note to lemma 6 that

$$\frac{y - y_0}{y - y_0 - 2a_0} = \gamma \prod_{\nu=0}^{n} \frac{R(x - x_\nu)}{R^2 - \bar{x}_\nu x}$$

where γ is a constant such that $| \gamma | = 1$, or

$$y = y_0 - 2(A - \Re y_0) \frac{\gamma R^{n+1} \prod_{\nu=0}^{n} \frac{x - x_\nu}{R^2 - \bar{x}_\nu x}}{1 - \gamma R^{n+1} \prod_{\nu=0}^{n} \frac{x - x_\nu}{R^2 - \bar{x}_\nu x}},$$

and conversely, the equality sign in I, II and III holds for all x such that $| x | < R$ when y is of the form indicated.

* This note has been changed to correspond to the changes in the note to lemma 6.

THEOREM 2. *Under the same assumptions as in theorem 1, we have*

$$|y - y_0| \leqq 2(A - \Re y_0)\frac{\kappa}{1 - \kappa}.$$

PROOF from theorem 1, I upon replacing the right-hand member a fortiori by $|y - y_0|\kappa + 2a_0\kappa$.*

NOTE 1. From the inequality in this theorem we may again derive some of the assumptions, viz., $y = y_0$ for $x = x_1, x_2, \cdots, x_n$.

NOTE 2. For $\kappa = \kappa_0$ we have

$$|y - y_0| \leqq 2(A - \Re y_0)\frac{R|x - x_0|}{|R^2 - \bar{x}_0 x| - R|x - x_0|},$$

and making $x_0 = 0$ in this inequality, we obtain a theorem due to Lindelöf (l.c.[11], p. 15)

$$|y - y(0)| \leqq 2(A - \Re y(0))\frac{|x|}{R - |x|},$$

whence Carathéodory's theorem (see above, p. 3) follows a fortiori.†

Assuming $x \neq x_0$, dividing the first formula above by $|x - x_0|$ and letting $x_0 \to x$, we find another theorem due to Lindelöf (l.c.)

$$\left|\frac{dy}{dx}\right| \leqq 2R\frac{A - \Re(y)}{R^2 - |x|^2},$$

to which we shall return later.

THEOREM 3. *Under the same assumptions as in theorem 1, and η being a constant, we have*

$$|y - \eta| \geqq \left|\eta - y_0 + 2(A - \Re y_0)\frac{\kappa^2}{1 - \kappa^2}\right| - 2(A - \Re y_0)\frac{\kappa}{1 - \kappa^2},$$

$$|y - \eta| \leqq \left|\eta - y_0 + 2(A - \Re y_0)\frac{\kappa^2}{1 - \kappa^2}\right| + 2(A - \Re y_0)\frac{\kappa}{1 - \kappa^2}.$$

THE PROOF follows a fortiori from theorem 1, III upon replacing the left member of the inequality by

$$\left|\eta - y_0 + 2a_0\frac{\kappa^2}{1 - \kappa^2}\right| - |y - \eta|.$$

* In a particular case, viz., when x is known to have such a value that $\Re(y) \geqq \Re(y_0)$, theorem 1, II gives the stronger inequality

$$|y - y_0| \leqq 2(A - \Re y_0)\frac{\kappa}{\sqrt{1 - \kappa^2}}.$$

† From theorem 2, it follows a fortiori that in the general case

$$|y| \leqq |y_0| + 2(A - \Re y_0)\frac{\kappa}{1 - \kappa} \leqq |\Im y_0| + 2A\frac{\kappa}{1 - \kappa} + |\Re y_0|\frac{1 + \kappa}{1 - \kappa},$$

a direct generalization of Carathéodory's theorem.

and

$$|y - \eta| - \left|\eta - y_0 + 2a_0 \frac{\kappa^2}{1 - \kappa^2}\right|$$

respectively.

NOTE 1. For $\kappa = 0$, both inequalities reduce to identities.

NOTE 2. For $\eta = y_0$, the lower inequality reduces to theorem 2.

NOTE 3. As a special case of theorem 3 we find for $\eta = 0$, $\kappa = \kappa_0$ and using the identity in lemma 2:

$$|y| \geqq \left|y_0 - 2(A - \Re y_0) \frac{R^2 |x - x_0|^2}{(R^2 - |x_0|^2)(R^2 - |x|^2)}\right| - 2(A - \Re y_0) \frac{R |x - x_0| |R^2 - \bar{x}_0 x|}{(R^2 - |x_0|^2)(R^2 - |x|^2)}$$

and

$$|y| \leqq \left|y_0 - 2(A - \Re y_0) \frac{R^2 |x - x_0|^2}{(R^2 - |x_0|^2)(R^2 - |x|^2)}\right| + 2(A - \Re y_0) \frac{R |x - x_0| |R^2 - \bar{x}_0 x|}{(R^2 - |x_0|^2)(R^2 - |x|^2)}.$$

For $x_0 = 0$ the second of these two formulas reduces to a theorem due to Lindelöf (ibid., written however in a simpler form),

$$|y| \leqq \left|y(0) - 2(A - \Re y(0)) \frac{|x|^2}{R^2 - |x|^2}\right| + 2(A - \Re y(0)) \frac{R|x|}{R^2 - |x|^2}.$$

Carathéodory's theorem follows a fortiori from this theorem also.

THEOREM 4. *Under the same assumptions as in theorem* 1, *and* η *being a constant such that* $\eta \neq \eta_0$ *and* $\Re(\eta) < A$,* *we have*

$$y \neq \eta \quad \textit{for} \quad \kappa = \kappa_0\kappa_1 \cdots \kappa_n < \left|\frac{y_0 - \eta}{y_0 + \bar{\eta} - 2A}\right|.\dagger$$

PROOF. We consider the function

$$\frac{y - \eta}{y + \bar{\eta} - 2A} = \frac{y - \eta}{-\eta - 2(A - \Re\eta)}$$

which is holomorphic for $|x| < R$ since $\Re(y + \bar{\eta} - 2A) < 0$ by our assumptions. Writing η for y_0 in the proof of theorem 1 we see at once that the function is < 1 in absolute value, and that it takes the value

$$\frac{y_0 - \eta}{y_0 + \bar{\eta} - 2A} \neq 0$$

* For $\Re(\eta) \geqq A$ we have $y \neq \eta$ in the entire given region $|x| < R$.

† This condition may also be written

$$\kappa^2 < \frac{|y_0 - \eta|^2}{|y_0 - \eta|^2 + 4(A - \Re y_0)(A - \Re\eta)}.$$

for $x = x_0, x_1, \cdots, x_n$. By reason of § 3 theorem 4 (in which we replace η by 0, y by the function just considered, and M by unity), our theorem is proved. (Another but less simple proof is obtained by applying the α method to the upper inequality in theorem 3. The details of this are left to the reader.)

Corollary. For $\kappa = \kappa_0$ we have

$$y \neq \eta \qquad \text{when} \qquad \left|\frac{R(x - x_0)}{R^2 - \bar{x}_0 x}\right| < \left|\frac{y_0 - \eta}{y_0 + \bar{\eta} - 2A}\right|,$$

or equivalently (by lemma 4, writing, x for u, x_0 for u_0, $\left|\frac{y_0 - \eta}{y_0 + \bar{\eta} - 2A}\right|$ for k, where $|(y_0 - \eta)/(y_0 + \bar{\eta} - 2A)|\,|x_0| < |x_0| < R$),

$$y \neq \eta \qquad \text{when} \qquad \left| x - x_0 \frac{R^2(|y_0 + \bar{\eta} - 2A|^2 - |y_0 - \eta|^2)}{R^2|y_0 + \bar{\eta} - 2A|^2 - |x_0(y_0 - \eta)|^2} \right|$$

$$< R \frac{(R^2 - |x_0|^2)\,|y_0 + \bar{\eta} - 2A|\,|y_0 - \eta|}{R^2|y_0 + \bar{\eta} - 2A|^2 - |x_0(y_0 - \eta)|^2},$$

and in particular

$$y \neq 0 \qquad \text{when} \qquad \left| x - x_0 \frac{R^2(|y_0 - 2A|^2 - |y_0|^2)}{R^2|y_0 - 2A|^2 - |x_0 y_0|^2} \right|$$

$$< R \frac{(R^2 - |x_0|^2)\,|y_0 - 2A|\,|y_0|}{R^2|y_0 - 2A|^2 - |x_0 y_0|^2}.$$

When we are satisfied with a smaller region, we have the simpler proposition:

$$y \neq \eta \qquad \text{when} \qquad |x - x_0| < \frac{(R^2 - |x_0|^2)\,|y_0 - \eta|}{R\,|y_0 + \bar{\eta} - 2A| + |x_0(y_0 - \eta)|},$$

as is readily found by replacing $|R^2 - \bar{x}_0 x|$ a fortiori by $R^2 - |x_0|^2 - |x_0(x - x_0)|$ in the condition given at the beginning.

Theorem 5. *Under the same assumptions as in theorem* 1, *we have*

$$-(A - \Re y_0)\frac{2\kappa}{1 - \kappa} \leqq \Re(y - y_0) \leqq (A - \Re y_0)\frac{2\kappa}{1 + \kappa}$$

or

$$-A\frac{2\kappa}{1 - \kappa} + \Re(y_0)\frac{1 + \kappa}{1 - \kappa} \leqq \Re(y) \leqq A\frac{2\kappa}{1 + \kappa} + \Re(y_0)\frac{1 - \kappa}{1 + \kappa}.$$

Proof. From theorem 1, II it follows a fortiori that

$$(\Re(y - y_0))^2 = (a_0 - a)^2 \leqq 4a_0 a\frac{\kappa^2}{1 - \kappa^2},$$

or equivalently $(a_0 - a)^2 \leqq (a_0 + a)^2\kappa^2$, or

$$\kappa \geqq \frac{|a - a_0|}{a + a_0},$$

or

$$\kappa \geqq \frac{a - a_0}{a + a_0} \geqq - \kappa, \qquad \text{or} \qquad a_0 \frac{1 - \kappa}{1 + \kappa} \leqq a \leqq a_0 \frac{1 + \kappa}{1 - \kappa},$$

which are equivalent to the inequalities to be proved.

NOTE 1. For $\kappa = 0$, both inequalities reduce to identities. Supposing the inequalities to be true for a certain function y, it follows conversely from the inequalities to the right that $\Re(y) = \Re(y_0)$ for $\kappa = 0$, and moreover we have a fortiori $\Re(y - y_0) < A - \Re(y_0)$ or $\Re(y) < A$, provided there exists an x_0 for which $\Re(y_0) < A$.

NOTE 2. For $\kappa = \kappa_0$ we have

$$- (A - \Re y_0) \frac{2R \,|\, x - x_0 \,|}{|\, R^2 - \bar{x}_0 x \,| - R \,|\, x - x_0 \,|} \leqq \Re(y - y_0)$$
$$\leqq (A - \Re y_0) \frac{2R \,|\, x - x_0 \,|}{|\, R^2 - \bar{x}_0 x \,| + R \,|\, x - x_0 \,|},$$

which reduces for $x_0 = 0$ to a formula given by Lindelöf (l.c.[11], p. 15):

$$- (A - \Re y(0)) \frac{2 \,|\, x \,|}{R - |\, x \,|} \leqq \Re(y - y(0)) \leqq (A - \Re y(0)) \frac{2 \,|\, x \,|}{R + |\, x \,|}.$$

COROLLARY. When $A' < \Re(y) < A$, then

$$(A' - \Re y_0) \frac{2\kappa}{1 + \kappa} \leqq \Re(y - y_0) \leqq (A - \Re y_0) \frac{2\kappa}{1 + \kappa}.$$

PROOF by applying the inequality to the right, which we know already from theorem 5, to the function $- y$, since $\Re(- y) < - A'$.

THEOREM 6. *Under the same assumptions as in theorem* 1, *we have*

$$|\, \Im(y - y_0) \,| \leqq 2(A - \Re y_0) \frac{\kappa}{1 - \kappa^2}.$$

PROOF follows at once from theorem 1, III by replacing a fortiori

$$y - y_0 + 2a_0 \frac{\kappa^2}{1 - \kappa^2}$$

on the left by its imaginary part.*

NOTE 1. For $\kappa = 0$ it follows from the inequality that $\Im(y) = \Im(y_0)$. The inequalities in theorems 5 and 6 taken together give again all the assumptions from which we started, viz., $y(x_\nu) = y(x_0)$ for $\nu = 1, 2, \cdots, n$ and $\Re(y) < A$, provided that there exists an x_0 for which $\Re(y_0) < A$.

* In the particular case when x is known to have such a value that $\Re(y) \geqq \Re(y_0)$, theorem 1, II gives the stronger inequality

$$|\, \Im(y - y_0) \,| \leqq 2(A - \Re y_0) \frac{\kappa}{\sqrt{1 - \kappa^2}}.$$

NOTE 2. For $\kappa = \kappa_0$, we have

$$|\mathfrak{J}(y - y_0)| \leqq 2(A - \Re y_0) \frac{R\,|x - x_0| \cdot |R^2 - \bar{x}_0 x|}{(R^2 - |x_0|^2)(R^2 - |x|^2)},$$

which gives, for $x_0 = 0$, another formula due to Lindelöf (l.c.):

$$|\mathfrak{J}(y - y(0))| \leqq 2(A - \Re y(0)) \frac{R\,|x|}{R^2 - |x|^2}.$$

THEOREM 7, CONCERNING THE DERIVATIVE OF y. *Under the same assumptions as in theorem* 1, *we have*

$$\left|\frac{dy}{dx}\right| \leqq 2R \frac{A - \Re(y)}{R^2 - |x|^2} \leqq 2R \frac{A - \Re(y_0)}{R^2 - |x|^2} \frac{1 + \kappa}{1 - \kappa}.\text{*}$$

$$\left|\frac{dy}{dx}\right| < 2R \frac{A - A'}{R^2 - |x|^2}.$$

PROOF. The first inequality is already known from note 2 to theorem 2. The second inequality follows by replacing $a = A - \Re(y)$ a fortiori by

$$a_0 \frac{1 + \kappa}{1 - \kappa} = (A - \Re y_0) \frac{1 + \kappa}{1 - \kappa}$$

in view of the last inequality in the proof of theorem 5.

NOTE 1. For $\kappa = 0$, the second and third members are identical; the second member may therefore be removed without making the formula less general.

NOTE 2.† To find when the equality sign holds in this theorem, let us assume that for $x = x_0$ and $y = y_0$ ($|x_0| < R$, $\Re(y_0) < A$) we have

$$\left|\frac{dy}{dx}\right|_0 = 2R \frac{A - \Re(y_0)}{R^2 - |x_0|^2}.$$

The function

$$u = \frac{y - y_0}{y - y_0 - 2a_0} \frac{R^2 - \bar{x}_0 x}{R(x - x_0)}$$

is holomorphic for $|x| < R$, and from theorem 1, I with $\kappa = \kappa_0$, we find $|u| \leqq 1$ for $|x| < R$. Moreover,

$$u(x_0) = \lim_{x \to x_0} u = -\frac{R^2 - |x_0|^2}{2R(A - \Re y_0)} \lim_{x \to x_0} \frac{y - y_0}{x - x_0} = -\frac{R^2 - |x_0|^2}{2R(A - \Re y_0)} \left(\frac{dy}{dx}\right)_0,$$

whence $|u(x_0)| = 1$. By the argument used in § 3, theorem 1, note 2, it

* When it is known moreover that $\Re(y) > A'$, we have a fortiori

† This note has been changed in the translation to correspond to the changes in the note to lemma 6.

follows that u is a constant $= -\gamma$, where $|\gamma| = 1$, and consequently

$$y = y_0 + 2\gamma(A - \Re y_0)\frac{R(x - x_0)}{R^2 - \bar{x}_0 x + \gamma R(x - x_0)}.$$

Conversely, it is seen that for any y of this form, where $|x_0| < R$, $\Re(y_0) < A$, $|\gamma| = 1$, the equality sign holds to the left in our theorem when $x = x_0$.

NOTE 3. Writing y as a power series in x

$$y(x) = c_0 + c_1 x + c_2 x^2 + \cdots,$$

(with a radius of convergence $\geqq R$), the first inequality in theorem 7 gives the relation

$$R|c_1| \leqq 2(A - \Re c_0)$$

for $x_0 = 0$, as noted by Lindelöf (l.c.). As in note 2 to theorem 5 in § 3 we may derive from this the more general relation

$$R^q|c_q| \leqq 2(A - \Re c_0),$$

q being a positive integer.

THEOREM 8, CONCERNING THE DIFFERENCE QUOTIENT OF y. *Under the same assumptions as in theorem* 1, *we have for* $x \neq x^*$

$$\left|\frac{y - y^*}{x - x^*}\right|^2 \leqq 4R^2\frac{(A - \Re y)(A - \Re y^*)}{(R^2 - |x|^2)(R^2 - |x^*|^2)},$$

x^* being a new independent variable, and $y^* \equiv y(x^*)$. When conversely this inequality is satisfied, and there exists a certain value of x for which $\Re(y) < A$, then $\Re(y) < A$ for all x.

PROOF. Theorem 1, II, states for $\kappa = \kappa_0$ that the inequality

$$|y - y_0|^2 \leqq 4a_0 a\frac{\kappa_0^2}{1 - \kappa_0^2} = |x - x_0|^2 \cdot 4R^2\frac{a_0 a}{(R^2 - |x|^2)(R^2 - |x_0|^2)}$$

is complete when an x_0 exists such that $\Re(y_0) < A$. Replacing x_0 by x^*, our theorem is proved.

NOTE. When it is desired to avoid y or y^* to the right in the inequality in theorem 8, we replace a fortiori (as in the preceding theorem)

$$A - \Re(y) \quad \text{by} \quad (A - \Re y_0)\frac{1 + \kappa}{1 - \kappa}$$

and

$$A - \Re(y^*) \quad \text{by} \quad (A - \Re y_0)\frac{1 + \kappa^*}{1 - \kappa^*}.$$

Exactly as in note 2 to theorem 6 in § 3, this may be varied in several ways.

FUNCTIONS OF LIMITED VARIATION IN AN INFINITE NUMBER OF DIMENSIONS.

By P. J. Daniell.

In a recent issue of these Annals appeared a paper by the author on "A General Form of Integral."* Integration was defined for functions of perfectly general elements, p. In a later paper (in the last issue), the author was able to define two kinds of integral in a space of a denumerably infinite number of dimensions. One of these was the generalization of the Lebesgue integral in the interval 0 to 1, the other an iterated Stieltjes integral of positive type. In the present paper functions, of a more general class, which are of limited variation in a denumerably infinite number of dimensions, are defined together with their corresponding Stieltjes integrals. References are made to Fréchet's thesis.† Our attention will be confined in this paper to a "finite domain" or interval

$$a_1 \leq x_1 \leq b_1, \quad \cdots, \quad a_n \leq x_n \leq b_n, \quad \cdots.$$

1. Introductory remarks. For the sake of definiteness we first define what we mean by a function of limited variation in a finite number of dimensions. Let $\alpha(x_1, x_2, \cdots x_n)$ be a function of a finite number of variables in the finite domain

$$(a_1 \leq x_1 \leq b_1, \cdots, a_n \leq x_n \leq b_n).$$

Divide the ranges of the variables into subranges as, for example, by the numbers

$$a_1 = x_{10} < x_{11} < \cdots < x_{1k} = b_1$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$a_n = x_{n0} < x_{n1} < \cdots < x_{nk} = b_n.$$

This divides the original interval into a finite number of rectangles or subintervals which we name by their extreme upper corners. We denote

$$\Delta_i\alpha = \alpha(x_{1i}, x_2, x_3 \cdots x_n) - \alpha(x_{1i-1}, x_2, \cdots, x_n),$$
$$\Delta_{ij}{}^2\alpha = \Delta_j(\Delta_i\alpha) = \alpha(x_{1i}, x_{2j}, x_3, \cdots, x_n)$$
$$- \alpha(x_{1i-1}, x_{2j}, x_3 \cdots x_n) - \alpha(x_{1i}, x_{2j-1}, x_3, \cdots, x_n)$$
$$+ \alpha(x_{1i-1}, x_{2j-1}, x_3, \cdots, x_n),$$

* P. J. Daniell, These Annals, vol. 19 (1918), p. 279. In this paper the author erroneously stated that E. H. Moore's integral is a special instance of his. He takes this opportunity to offer his apologies to Professor Moore and to his readers.

† M. Fréchet, Rendiconti di Circolo matematico di Palermo, vol. 22 (1906), p. 40.

and so on; finally

$$\Delta_{ij}{}^{n}{}_{\cdots q}\alpha = \alpha(x_{1i}, x_{2j}, \cdots, x_{nq}) - \cdots + \cdots$$
$$\cdots + (-1)^n \alpha(x_{1i-1}, x_{2j-1}, \cdots, x_{nq-1}).$$

DEFINITION 1. 1. *Positive type.* The function $\alpha(x_1, \cdots, x_n)$ is said to be of positive type in the variables $(x_1, \cdots, x_n)$, if

$$\Delta_{ij}{}^{n}{}_{\cdots q}\alpha \geqq 0$$

for all subintervals $(i, j, \cdots, q)$ whatever.

DEFINITION 1. 2. *Limited variation.* The function $\alpha(x_1, \cdots, x_n)$ is said to be of limited variation in the variables $(x_1, \cdots, x_n)$ if

$$\sum_{ij\cdots q} |\Delta_{ij}{}^{n}{}_{\cdots q}\alpha| \leqq K,$$

where K is some finite number independent of the manner in which the interval is divided. The lower bound of such numbers K is called the total variation of α in the interval $(a_1, a_2, \cdots, a_n)$ to $(b_1, b_2, \cdots, b_n)$.

DEFINITION 1. 3. *Simple type.* The function $\alpha(x_1, \cdots, x_n)$ is said to be of simple type in the interval $(a_1, a_2, \cdots, a_n)$ to $(b_1, b_2, \cdots, b_n)$, if it is zero whenever any one of the variables, x_r, attains its lower limit, a_r.

THEOREM 1. 4. *A function which is of positive and simple type in the variables $(x_1, \cdots, x_n)$ is also of positive type in any collection of variables chosen from among $(x_1, \cdots, x_n)$, the remainder being held constant.*

Let $x_p, x_q, \cdots, x_t$, be the m variables chosen and consider an interval $(c_p, c_q, \cdots, c_t)$ to $(x_p, x_q, \cdots, x_t)$. The nth order difference of α in the interval

$$(a_1, a_2, \cdots c_p, \cdots, c_t, \cdots a_n) \text{ to } (x_1, \cdots, x_n),$$

that is to say where $a_1, \cdots, a_n$ are the lower limits for all variables except the $p, q, \cdots t$ variables,

$$\Delta\alpha^n = \alpha(x_1, \cdots, x_n) - \cdots + \cdots + (-1)^n\alpha(a_1, \cdots, c_p, \cdots, a_n)$$
$$= \alpha(x_1, \cdots, x_n) - \cdots + \cdots + (-1)^m\alpha(x_1, \cdots, c_p, \cdots, c_t, \cdots, x_n).$$

For, since α is of simple type, the remaining values of α all vanish.

[Example. In the interval (a_1, c_2) to (x_1, x_2)

$$\begin{aligned}\Delta^2\alpha(x_1x_2) &= \alpha(x_1, x_2) - \alpha(x_1, c_2) - \alpha(a_1, x_2) + \alpha(a_1, c_2)\\ &= \alpha(x_1, x_2) - \alpha(x_1, c_2) - 0 + 0\\ &= \Delta\alpha(x_1, x_2)\end{aligned}$$

in the interval c_2 to x_2, x_1 being constant.]

Then the mth order difference of α in the interval $(c_p, c_q, \cdots c_t)$ to $(x_p, x_q, \cdots x_t)$, the other variables $x_1, \cdots, x_n$ being held constant, is equal to the nth order difference of α in the interval

$$(a_1, \cdots, c_p, \cdots, c_t, \cdots, a_n) \text{ to } (x_1, \cdots, x_n).$$

If α is of positive type, the latter nth order difference is non-negative, or the former mth order difference is also non-negative. The theorem is proved.

THEOREM 1. 5. *If $\alpha(x_1, \cdots, x_n)$ is of limited variation and simple type in the variables $(x_1, \cdots, x_n)$, it is also of limited variation in any collection of variables chosen from among $(x_1, \cdots, x_n)$, the remainder being held constant. Also, if the* nth *order total variation $\leq K$, the latter variation is also $\leq K$.*

Let the interval $(a_p, a_q, \cdots, a_t)$ to $(b_p, b_q, \cdots, b_t)$ be subdivided into subintervals in any manner, one of these subintervals being

$$(c_p, c_q, \cdots, c_t) \text{ to } (d_p, d_q, \cdots, d_t).$$

By the reasoning employed in Theorem 1. 4, the multiple order difference of α in this interval is the same as the nth order difference of α in the interval

$$(a_1, \cdots, c_p, \cdots, c_t, \cdots, a_n) \text{ to } (x_1, \cdots, d_p, \cdots, d_t, \cdots, x_n).$$

The collection of subintervals of the latter type form a possible mode of subdivision of the n-dimensional interval and, therefore,

$$\Sigma \, | \Delta^m \alpha | \leq K,$$

if K is the total variation of α in the variables $(x_1, \cdots, x_n)$, in the interval $(a_1, \cdots, a_n)$ to $(b_1, \cdots, b_n)$. This proves the theorem.

[Example. Consider the variation of $\alpha(\xi_1, x_2)$ in the interval, $a_1 \leq \xi_1 \leq b_1$. Divide this interval by the numbers

$$a_1 = l_0 < l_1 < \cdots < l_n = b_1,$$

$$\alpha(l_r, x_2) - \alpha(l_{r-1}, x_2) = \alpha(l_r, x_2) - \alpha(l_r, a_2) - \alpha(l_{r-1}, x_2) + \alpha(l_{r-1}, a_2).$$

$$\sum_{r=1}^{n} | \Delta_r \alpha | = \sum_{r=1}^{n} | \Delta^2_{r, x_2} \cdot \alpha |$$

$$\leq K.]$$

2. **Infinite number of dimensions.**

DEFINITION 2. 1. *Positive type.* A function $\alpha(x_1, \cdots, x_n, \cdots)$ of the denumerably infinite number of variables $(x_1, x_2, \cdots, x_n, \cdots)$ is said to be of positive type, if, when $x_{n+1}, x_{n+2}, \cdots$ are held constant, it is of positive type in $(x_1, \cdots, x_n)$ for all integers n.

DEFINITION 2. 2. *Limited variation.* A function $\alpha(x_1, \cdots, x_n, \cdots)$ of the denumerably infinite number of variables $(x_1, \cdots x_n, \cdots)$ is said to be of limited variation, if, when $x_{n+1}, x_{n+2}, \cdots$ are held constant, it is of limited variation in $(x_1, \cdots, x_n)$ for all integers n, *these total variations being limited in their set.*

DEFINITION 2. 3. *Simple type.* A function $\alpha(x_1, \cdots, x_n, \cdots)$ is said to be of simple type in the interval

$$a_1 \leqq x_1 \leqq b_1, \cdots, a_n \leqq x_n \leqq b_n, \cdots,$$

if it is zero whenever any one of the variables, x_n, attains its lower limit, a_n.

THEOREM 2. 4. *Any function of limited variation and simple type is the difference of two limited functions of simple and positive type. Conversely the difference of two limited functions of simple and positive type is a function of limited variation and simple type.*

The second part of the theorem is easy to prove; for, if $\alpha = \beta - \beta'$,

$$\Delta^n\alpha = \Delta^n\beta - \Delta^n\beta',$$

$$|\Delta^n\alpha| \leqq \Delta^n\beta + \Delta^n\beta'$$

$$\Sigma\,|\Delta^n\alpha| \leqq \beta(b_1, b_2, \cdots, b_n, \cdots) + \beta'(b_1, b_2, \cdots, b_n, \cdots).$$

Also when $\beta = 0$, $\beta' = 0$, then $\alpha = 0$.

The proof of the first part is not so simple. Let K denote the upper bound of all variations of α, varying $(x_1, \cdots, x_n)$, holding $x_{n+1}, \cdots$, constant for all integers n and all $(x_{n+1}, \cdots)$ in the interval. By definition such a finite K exists. Denote by

$$\alpha_n(x_1, \cdots, x_n, x_{n+1}, \cdots)$$

the total variation of $\alpha(\xi_1, \xi_2, \cdots \xi_n, x_{n+1}, \cdots)$ holding $(x_{n+1}, x_{n+2}, \cdots)$ constant and varying $(\xi_1, \cdots, \xi_n)$ in the interval $(a_1, a_2, \cdots, a_n)$ to $(x_1, \cdots, x_n)$. This function is numerically not greater than K and is of simple type in all the variables. It is also of positive type in the variables $(x_1, \cdots, x_n)$, when $x_{n+1}, \cdots$, are regarded as parametric constants. By Theorem 1. 4. if $m \leqq n$, α_n is of positive type in the variables $(x_1, \cdots, x_m)$. Let

$$2\beta_n = \alpha + \alpha_n. \qquad 2\beta_n' = \alpha_n - \alpha.$$

Then β_n, β_n' are also not greater than K and are of simple type in all the variables. They are also of positive type in the variables $(x_1, \cdots, x_n)$, in fact they are the positive and negative variation functions of α in the n variables such that

$$\beta_n - \beta_n' = \alpha,$$

$$\beta_n + \beta_n' = \alpha_n,$$

the total variation function of α in the variables $(x_1, \cdots, x_n)$. By Theorem **1.** 5 if $m \leq n$,

$$\alpha_m \leq \alpha_n.$$

Then the sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{\beta_n'\}$ are nondecreasing sequences limited by K and possess limits ω, β, β', such that

$$\alpha = \beta - \beta', \qquad \beta + \beta' = \omega.$$

DEFINITION **2.** 5. We define the total variation function,

$$\omega(x_1, x_2, \cdots, x_n, \cdots) = \lim_{n \doteq \infty} \alpha_n(x_1, \cdots, x_n, \cdots),$$

as the limit, as n increases indefinitely, of the total variation function, α_n, of α in the variables $(x_1, \cdots, x_n)$, holding $x_{n+1}, \cdots$ constant. Similarly we can call β, β', the positive and negative variation functions respectively. Since β_n, β_n' are of positive and simple type in $(x_1, \cdots, x_n)$, by Theorem **1.** 4 if $m \leq n$, they are also of positive type in the variables $(x_1, \cdots, x_m)$. Therefore, in the limit, β, β' are of positive type in the variables $(x_1, \cdots, x_m)$ for all integers m; that is to say β, β' are of positive type in the sense of Def. **2.** 1. Again if $x_r = a_r$, $\beta_n = 0$, $\beta_n' = 0$ for all n, or

$$\beta = 0, \qquad \beta' = 0.$$

Hence β, β' are of simple type, and they are not greater than K. The theorem is proved.

3. **Generalized Stieltjes integral.** Let $\beta(x_1, \cdots, x_n, \cdots)$ be a limited function of simple and positive type in the interval

$$(a_1, \cdots, a_n, \cdots) \text{ to } (b_1, \cdots, b_n, \cdots).$$

as in our previous paper on integrals in an infinite number of dimensions, we choose the class T_0 to be the class of functions of a finite number of the variables, continuous in the given interval. As before T_0 is closed with respect to the operations of multiplication by a constant, addition and taking the modulus; the functions are bounded, and if $f_1(p), \cdots, f_n(p), \cdots$ are functions of class T_0 such that

$$f_1(p) \geq f_2(p) \geq \cdots \geq f_n(p) \geq \cdots \geq 0 = \lim_{n \doteq \infty} f_n(p)$$

for all p in the interval; then

$$\lim_{n \doteq \infty} \max_p f_n(p) = 0.$$

DEFINITION **3.** 1. Let $f(x_p, \cdots, x_t)$ be a function of class T_0, then we define the integral of f with respect to β in the given interval as

$$\int f d\beta = \int_{a_p}^{b_p} \cdots \int_{a_t}^{b_t} f(x_p, \cdots x_t) dB(x_p, \cdots x_t),$$

where the integration is over the finite number of variables $(x_p, \cdots, x_t)$ and where

$$B(x_p, \cdots, x_t) = \beta(b_1, \cdots, x_p, \cdots, x_t, \cdots, b_n),$$

all the variables except $(x_p, \cdots, x_t)$ being made equal to their upper bounds $(b_1, \cdots, b_n, \cdots)$. The right-hand integral is definite because by Theorem **2.** 4, and Definition **2.** 1, $B(x_p, \cdots, x_t)$ is a limited function of positive type and f is a continuous function of the finite number of variables $(x_p, \cdots, x_t)$.

It happens that a continuous function of m variables can also be regarded as a continuous function of $m + 1$, $m + 2$, $\cdots$ variables, by subjoining variables with respect to which the function is constant. On this account it is necessary to prove that our definition of the integral is self-consistent. It is sufficient if we consider the case where one variable is added; that is, if x_u is any variable other than x_p, x_q, $\cdots$, x_t, then we can prove that, if $f(x_p, \cdots, x_t)$ is a continuous function,

$$\int f(x_p, \cdots, x_t) dB(x_p, \cdots, x_t) = \int f(x_p, \cdots, x_t) dB(x_p, \cdots, x_t, x_u).$$

[For convenience we have placed the variable x_n last, but it may occur anywhere with respect to $x_p, \cdots, x_t$.] The right-hand integral is found as the limit of

$$\sum_{\alpha, \ldots, \theta, \eta} f(\xi_{p\alpha}, \cdots, \xi_{t\theta}) \Delta^{x_{p,\alpha+1} \cdots x_{t,\theta+1} x_{u,\eta+1}}_{x_{p\alpha} \cdots x_{t\theta} x_{u\eta}} B(x_p \cdots x_t, x_u)$$
$$= \sum_{\alpha, \ldots, \theta} f(\xi_{p\alpha}, \cdots, \xi_{t\theta}) \Delta^{x_{p,\alpha+1} \cdots x_{t,\theta+1}}_{x_{p\alpha} \cdots x_{t\theta}} \sum \Delta^{x_{u,\eta+1}}_{x_{u\eta}} B(x_p \cdots x_t x_u).$$

But

$$\sum_\eta \Delta^{x_{u,\eta+1}}_{x_{u\eta}} B(x_p, \cdots, x_t x_u) = B(x_p, \cdots, x_t, b_u) - B(x_p, \cdots, x_t, a_u)$$
$$= B(x_p, \cdots, x_t) - 0.$$

This proves the consistency of the integral definition. It follows without further difficulty that the integral

$$I(f) = \int f d\beta$$

satisfies the postulates,

$$(C) \qquad I(cf) = cI(f),$$

$$(A) \qquad I(f_1 + f_2) = I(f_1) + I(f_2),$$

$$(P) \qquad I(f) \geqq 0, \text{ if } f \geqq 0 \text{ for all elements } p.$$

Also if

$$B = \beta(b_1, \cdots, b_n, \cdots)$$

$$|I(f)| \leqq B \max_p |f(p)|.$$

Using the fact stated in the paragraph before Definition **3.** 1, we observe

that postulate (L) is also satisfied, that is to say, if

$$f_1(p) \geqq f_2(p) \geqq \cdots \geqq f_n(p) \cdots \geqq 0 = \lim f_n(p)$$

for all elements p, then

$$\lim_{n=\infty} I(f_n) \leqq B \lim_{n=\infty} \max_p f(p)$$
$$= 0.$$

DEFINITION **3. 2.** Let $\alpha(x_1, \cdots, x_n, \cdots)$ be a function of limited variation and simple type. By Theorem **2. 4** it is the difference of two functions β, β' limited and of simple and positive type. We define

$$S(f;\ \alpha) = I(f;\ \beta) - I(f;\ \beta'),$$

$$\int f d\alpha = \int f d\beta - \int f d\beta'.$$

$S(f)$ satisfies postulates (C) (A) (L). Moreover

$$|S(f;\ \alpha)| \leqq (B + B') \max |f|,$$

that is to say, it also satisfies postulate M where

$$M(f) = \max f.$$

As we proved in the previous paper any continuous function is the limit of a sequence of functions of class T_0 limited in their set and therefore we can define $S(f;\ \alpha) = \int f d\alpha$ for all continuous functions f. We can also extend the definition of the Stieltjes integral to a wider class of summable functions by the methods used in our article "A General Form of Integral."

4. *Example* 1. Let $\alpha_i(x_i)$, $\alpha_{ij}(x_i, x_j)$, $\cdots$ be functions of simple type and limited variation in the variables specified, in the interval

$$(a_1, \cdots, a_n, \cdots) \text{ to } (b_1, \cdots, b_n, \cdots);$$

and let V_i, V_{ij}, $\cdots$ be the corresponding total variations. Then if

$$\sum_{n=1}^{\infty} |k_i| V_i = S_1, \qquad \sum_{i,j} |k_{ij}| V_{ij} = S_2, \cdots$$

are convergent series and if

$$\sum_{n=1}^{\infty} S_n = K$$

is convergent, we may define

$$\alpha(x_1, \cdots, x_n, \cdots) = \sum_i k_i \alpha_i(x_i) + \sum_{i,j} k_{ij} \alpha_{ij}(x_i, x_j) + \cdots,$$

and this α will be of simple type and limited variation. In particular choose the interval from

$$(0, 0, 0, \cdots) \text{ to } (b_1, b_2, \cdots, b_{n!}, \cdots)$$

and let

$$\alpha(x_1, x_2, \cdots, x_n, \cdots) = \sum_{n=1}^{\infty} \frac{1}{n_4}\left[\prod_{s=1}^{n} \varphi(x_s)\right],$$

$\varphi(x)$ = integral part of $x \div (1 + x^2)$. $\varphi(x)$ is of limited variation from 0 to b_s, in fact from 0 to ∞, for, at $x = n$, it makes a sudden increase from

$$\frac{n-1}{1+n^2} \quad \text{to} \quad \frac{n}{1+n^2},$$

and then decreases continually from

$$\frac{n}{1+n^2} \quad \text{to} \quad \frac{n}{1+(n+1)^2},$$

when it suddenly increases again and so on. The variation of $\varphi(x)$ from 0 to ∞ is

$$V = \sum_{n=1}^{\infty} \frac{1}{1+n^2} + \frac{2n^2+n}{(n^2+2n+2)(n^2+1)} < 3\Sigma \frac{1}{1+n^2}$$

and therefore is finite.

The variation of α in the variables $(x_1, \cdots, x_n)$ or

$$\alpha_n < V^n\left(\frac{1}{n!} + \frac{1}{(n+1)!} + \cdots\right)$$

for $|\varphi(x)| < 1$.

$$\alpha_n < \frac{V^n}{(n-1)!}.$$

The right-hand expression has a finite upper bound, which proves that α is of limited variation, though, since this is only a particular case of the general example, we can see that the total variation is less than

$$\sum_{n=1}^{\infty} \frac{1}{n!} V^n = e^V - 1.$$

Example 2. Of an entirely different type is the following example. Let (x_n) denote the lower bound of the variables $(x_1, x_2, \cdots, x_n, \cdots)$ and let the interval be from $(0, 0, 0, \cdots)$ to $(3, 3, 3, \cdots)$; then

$$\begin{aligned} \alpha(x_1, \cdots, x_n, \cdots) &= 0 \qquad & 0 \leq (x_n) < 1 \\ &= 1 & 1 \leq (x_n) < 2 \\ &= -1 & (x_n) \geq 2, \end{aligned}$$

is of limited variation and simple type. To prove this it is sufficient to observe that we can express α as the difference of two limited functions of simple and positive type, in fact,

$$\alpha = \beta - 2\beta'',$$

where

$$\beta(x_1, \cdots, x_n, \cdots) = 0, \qquad 0 \leqq (x_n) < 1,$$
$$= 1, \qquad (x_n) \geqq 1.$$
$$\beta''(x_1, \cdots, x_n, \cdots) = 0, \qquad 0 \leqq (x_n) < 2,$$
$$= 1, \qquad (x_n) \geqq 2.$$

Consider the nth order difference of β in an interval

$$(c_1, c_2, \cdots, c_n) \text{ to } (x_1, x_2, \cdots, x_n)$$

the variables $x_{n+1}, x_{n+2}, \cdots$ being held constant. Since $c_r \leqq x_r$, if any $x_r < 1$, $\beta = 0$ at every vertex of the "interval" and the nth order difference is also 0. We suppose then that $(x_n) \geqq 1$. Let s of the c's be $\geqq 1$, the rest being < 1. Then at the extreme upper vertex $\beta = 1$, at s of the vertices next to this $\beta = 1$ and at the others $\beta = 0$, at $\binom{s}{2}$ of the next vertices $\beta = 1$ and at the others $\beta = 0$, and so on. Then the interval difference

$$\Delta^n\beta = 1 - \binom{s}{1} + \binom{s}{2} - \cdots + (-1)^s$$
$$= (1 - 1)^s$$
$$= 0.$$

If every $c_r < 1$, $\beta = 0$ at every vertex except the extreme upper vertex, at which $\beta = 1$. Hence the interval difference of β is always zero except when $(x_n) \geqq 1$ and $c_r < 1$, $(r = 1, \cdots, n)$, and in that case the interval difference $= 1$.

In particular, β is of positive type. Similarly it can be shown that β'' is of positive type.

It also follows that, if $f(x_p, \cdots, x_t)$ is a function of class T_0,

$$\int f d\beta = f(1, 1, \cdots, 1)$$

$$\int f d\beta'' = f(2, 2, \cdots, 2),$$

so that

$$\int f d\alpha = f(1, 1, \cdots, 1) - 2f(2, 2, \cdots, 2).$$

Lastly if f is any continuous function of the variables $(x_1, \cdots, x_n, \cdots)$,

$$\int f d\alpha = f(1, 1, \cdots, 1, \cdots) - 2f(2, 2, \cdots, 2, \cdots).$$

The reader can readily invent other examples of this type.

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A NEW SEQUENCE OF INTEGRAL TESTS FOR THE CONVERGENCE AND DIVERGENCE OF INFINITE SERIES.*

By Raymond W. Brink.

Introduction. The convergence or divergence of a given series is determined by the behavior of its partial sum, the sum of the first n terms of the series. If this partial sum approaches a finite limit as n increases indefinitely, the series is said to converge; otherwise it diverges.

For certain series, as for example a geometric series, it is possible to find explicitly a simple analytic expression for a function that for successive positive integral values of the variable takes on the values of the corresponding partial sums. Whenever such a function is known, the series is known to converge if the function approaches a finite limit when the variable becomes infinite, and to diverge or oscillate otherwise.

Often when an ordinary examination of a given series $u_0 + u_1 + \cdots$, brings to light no such function $S(x)$ having the property that

$$S(n) = S_n = u_0 + u_1 + u_2 + \cdots + u_{n-1},$$

one can readily find a simple function $u(x)$ such that $u(n) = u_n$. And in other cases where no simple function $S(x)$ and no simple function $u(x)$ are apparent, it is possible to find a simple form for a function $r(x)$ having the property that $r(n) = r_n = u_{n+1}/u_n$, the general test-ratio of the series.

More generally, it is often possible to find some simple function $f(x)$ such that

$$f(n) = f_n = \varphi(u_n, u_{n+1}, \cdots, u_{n+k}),$$

a function of the nth term of the series and of a certain number of the following terms. Since $u_n = S_{n+1} - S_n$, we can write

$$f(n) = f_n = \psi(S_n, S_{n+1}, \cdots, S_{n+k+1}).$$

Then the equation

$$f(x) = \psi(S(x), S(x+1), \cdots, S(x+k))$$

is a difference equation in $S(x)$, which, if $S(x)$ is properly defined in some initial interval, defines the function $S(x)$ elsewhere so that $S(n) = S_n$. An examination of this function $S(x)$ is then often sufficient to establish the convergence or divergence of the series in question.

* The results of this paper were presented to the American Mathematical Society, April, 1916. Many of the proofs, however, have been worked out in their present form since that time.

By means of this principle the following sequence of integral tests were suggested. Other tests may be suggested in a similar way. In discovering the form of a test involving a certain relation between terms of a series, instead of setting up the difference equation for $S(x)$ it is often simpler to set up a difference equation for a function involved in an earlier test, and then make use of the form of the earlier test. Thus the form of the test of § 2 is obtained from that of § 1. And from the form of the former test may be obtained that of § 4. Whether the form is discovered in this way or by setting up the difference equation in $S(x)$, which may always be done, the difference equations will be replaced by differential equations whose solutions will serve equally well to suggest the tests. The validity and usefulness of the tests suggested in this way will be examined by independent methods.

Throughout the paper the symbol $\{u_n\}$ will be used to denote a sequence u_0, u_1, u_2, etc., whose general term u_n is positive. Since a finite number of terms do not affect questions of convergence, zero will be used as the lower limit of summation or integration in stating theorems. It seems better to omit certain changes in phraseology that suggest themselves for cases in which the conditions of a theorem apply only for values of a variable from a certain number on.

1. **The Maclaurin-Cauchy Integral Test.** Suppose we are given the series

$$u_0 + u_1 + u_2 + \cdots.$$

If $u(x)$ is a function such that $u(n) = u_n$, the difference equation

$$S(x+1) - S(x) = u(x) \tag{1}$$

defines a function $S(x)$ such that

$$S(n) = S_n = u_0 + u_1 + \cdots + u_{n-1},$$

provided that $S(x)$ is properly defined in some initial interval. From equation (1), by the mean value theorem,

$$S'(\xi) = u(x) \tag{2}$$

where $x < \xi < x + 1$, and

$$S'(x) = \frac{dS}{dx}.$$

We may reasonably hope that under suitable restrictions, $S'(\xi)$ may for our purposes be replaced by $S'(x)$ and that we may write, approximately,

$$S(x) = C + \int_0^x u(x)dx.$$

We may, then, hope that if certain restrictions are placed upon $u(x)$, $S(x)$

will approach a limit or will fail to do so, and the given series will therefore converge or diverge according as the integral

$$\int_0^\infty u(x)dx$$

converges or diverges. Thus in this first application of our method is suggested the Maclaurin-Cauchy test* which is essentially as follows:

Given a sequence $\{u_n\}$. *Let* $u(x)$ *be a positive integrable function having the property that* $u(n) = u_n$, *and that either*

(α) *a constant* k *exists such that when* $|x - x'| \leqq 1$,

$$\frac{u(x)}{u(x')} < k,$$

or

(β) *a constant* m *exists, positive or zero, such that, when* $x' \geqq x + m$,

$$u(x) \geqq u(x').$$

A necessary and sufficient condition for the convergence of the series $\sum_{n=0}^{\infty} u_n$ *is the convergence of the integral* $\int_0^\infty u(x)dx$. The method of establishing this test and its many applications are too well known to require consideration here.

This test may be extended to test series of positive and negative or complex terms, as follows.

THEOREM I. *Given a series*

$$u_0 + u_1 + u_2 + \cdots.$$

Let $u(x)$ *be an integrable function of the real variable* x *such that*

$$u(n) = u_n, \tag{1}$$

$$\lim_{x=\infty} u(x) = 0, \tag{2}$$

$$|u(x) - u_n| \leqq v_n, \qquad 0 \leqq x - n \leqq 1, \tag{3}$$

the series $\sum_{n=\mu}^{\infty} v_n$ *being a convergent series. A necessary and sufficient condition for the convergence of the series* $\sum_{n=0}^{\infty} u_n$ *is the convergence of the integral* $\int_0^\infty u(x)dx$.

Proof. If we write

$$S_\mu^n = S_n - S_\mu = u_\mu + u_{\mu+1} + \cdots + u_{n-1},$$

* Maclaurin: Treatise on Fluxions, I, p. 289, 1742. Cauchy: Exercises de Mathématiques, vol. 2, p. 221, 1827. Oeuvres complètes, 2e série, vol. VII, p. 267.

we have

$$\left|\int_\mu^n u(x)dx - S_\mu^n\right| = \left|\int_\mu^{\mu+1}(u(x) - u_\mu)dx + \int_{\mu+1}^{\mu+2}(u(x) - u_{\mu+1})dx + \cdots + \int_{n-1}^{n}(u(x) - u_{n-1})dx\right|$$

$$\leqq \int_\mu^{\mu+1}|u(x) - u_\mu|dx + \int_{\mu+1}^{\mu+2}|u(x) - u_{\mu+1}|dx + \cdots + \int_{n-1}^{n}|u(x) - u_{n-1}|dx$$

$$\leqq v_\mu + v_{\mu+1} + \cdots + v_{n-1} \leqq \sum_{n=\mu}^{\infty} v_n.$$

Since $\lim_{x=\infty} u(x) = 0$, $\lim_{x=\infty}\int_n^x u(x)dx = 0$, if $n \leqq x \leqq n + 1$. Therefore

$$\lim_{x=\infty}\left|\int_\mu^x u(x)dx - S_\mu^n\right| \leqq \sum_{n=\mu}^{\infty} v_n, \qquad \text{where} \qquad n \leqq x \leqq n + 1.$$

This inequality establishes the theorem, since by increasing μ we can make $\sum_{n=\mu}^{\infty} v_n$ as small as we wish. Moreover the inequality provides us with limits on the value of the series if the series converges, or on its oscillation if it oscillates.

Hardy* gives the following theorem: *If* (1) $u(x)$ *possesses a continuous derivative* $u'(x)$, (2) $\lim_{x=\infty} u(x) = 0$ *and* (3) *the integral* $\int_0^\infty |u'(x)|dx$ *converges, the series* $\sum_{n=0}^{\infty} u(n)$ *converges, oscillates, or diverges together with the integral* $\int_0^\infty u(x)dx$. As Hardy points out, this theorem includes a useful test due to Bromwich.† On the other hand, Hardy's test is contained in Theorem I above. For suppose that the integral $\int_0^\infty |u'(x)|dx$ converges. We have

$$|u(x) - u_n| = \left|\int_n^x u'(x)dx\right| \leqq \int_n^{n+1}|u'(x)|dx, \qquad 0 \leqq x - n \leqq 1.$$

Then since $\sum_{n=0}^{\infty}\int_n^{n+1}|u'(x)|dx$ converges, $v_n = \int_n^{n+1}|u'(x)|dx$ satisfies the conditions of Theorem I. Bromwich and Hardy give some interesting examples of series of complex terms tested by these tests. For most applications the tests of Hardy and Bromwich and that of Theorem I are equally useful.

* Hardy: Proceedings of the London Math. Society, 2d ser., vol. 9 (1910), p. 127.

† Bromwich: Proceedings of the London Math. Society, 2d ser., vol. 6 (1908), p. 329.

2. Integral Tests of the Second Kind. Given a sequence $\{u_n\}$. Let $r_n = u_{n+1}/u_n$ be the *ratio* of the series $\sum_{n=0}^{\infty} u_n$. Then

$$\log r_n = \log u_{n+1} - \log u_n.$$

If $r(x)$ is a positive continuous function such that $r(n) = r_n$ and if $u(x)$ is properly defined in some initial interval, the difference equation

$$\log r(x) = \log u(x+1) - \log u(x)$$

defines a function $u(x)$ elsewhere so that $u(n) = u_n$. By the mean-value theorem

$$\log r(x) = \frac{u'(\xi)}{u(\xi)}$$

where $x < \xi < x + 1$. We may reasonably hope that under suitable restrictions on $r(x)$, $u'(\xi)/u(\xi)$ will differ from $u'(x)/u(x)$ by magnitudes of a low order for large values of x, and that without excessive error we may write

$$u(x) = ke^{\int_0^x \log r(x)dx}.$$

Using this form for $u(x)$ in the Maclaurin-Cauchy test, we obtain the integral

$$\int_0^{\infty} e^{\int_0^x \log r(x)dx} dx$$

as a form which under certain conditions may be expected to converge or diverge with the given series. In obtaining this form directly from the difference equation in $S(x)$ we are led to the differential equation

$$\frac{S''(x)}{S'(x)} = \log r(x),$$

whose solution is

$$S(x) = c + k\int_0^x e^{\int_0^x \log r(x)dx} dx.$$

The tests suggested in this way are much the most important of the present sequence of tests except for the Maclaurin-Cauchy test. These tests and others arising from them, as well as certain applications, have been treated by the writer in another paper where they were called *integral tests of the second kind*.* The three following theorems are quoted from that article.

THEOREM II. *Given the sequence* $\{u_n\}$. *Let* $r_n = u_{n+1}/u_n$, *and let* $r(x)$ *be a positive, integrable function having the property that* $r(n) = r_n$, *and that a*

* Transactions of the American Math. Society, vol. 19 (1918), p. 186.

constant m exists, positive or zero, such that $r(x') \geqq r(x)$ when $x' \geqq x + m$. A necessary and sufficient condition for the convergence of the series $\sum_{n=0}^{\infty} u_n$ is the convergence of the integral

$$\int_0^\infty e^{\int_0^x \log r(x)dx} dx.$$

THEOREM III. *Given the sequence $\{u_n\}$. Let $r(x)$ be a function with an integrable derivative $r'(x)$, such that $r(n) = r_n = u_{n+1}/u_n$. If*

$$0 < A \leqq r(x) \leqq B,$$

and the integral

$$\int_0^\infty |r'(x)| dx$$

converges, a necessary and sufficient condition for the convergence of the series $\sum_{n=0}^{\infty} u_n$ is the convergence of the integral

$$\int_0^\infty e^{\int_0^x \log r(x)dx} dx.$$

THEOREM IV. *Given the sequence $\{u_n\}$. Let $r_n = u_{n+1}/u_n$, and let $r(x)$ be a positive, integrable function satisfying the preliminary conditions of one of the Theorems II or III, and the further condition that*

$$|r(x) - 1| < a < 1, \qquad \mu_1 < x.$$

A sufficient condition for the convergence of the series $\sum_{n=0}^{\infty} u_n$ is the convergence of the integral

$$\int_0^\infty e^{\int_0^x (r(x)-1)dx} dx.$$

This condition is also necessary for the convergence of the series if from a certain point on

$$|r(x) - 1| < \frac{k}{x},$$

where k is some constant.

3. The Second-Difference Test. Suppose that we are given a sequence $\{u_n\}$ and a function $D(x)$ satisfying the condition that

$$D(n) = D_n = u_n - u_{n+1}.$$

By a method similar to that used in the last section we are led to an approximate form

$$u(x) = C - \int_0^x D(x)dx$$

for a function $u(x)$ such that $u(n) = u_n$. The integral in the Maclaurin-Cauchy test will clearly be of most interest if

$$C = \int_0^\infty D(x)dx.$$

Assigning this value to C, we are led to hope that under certain restrictions the given series will converge or diverge together with the integral

$$\int_0^\infty \int_x^\infty D(x)dxdx.$$

We have the following theorem.

THEOREM V. *Given the sequence* $\{u_n\}$ *where* $\lim_{n=\infty} u_n = 0$, *and a positive, integrable function* $D(x)$ *such that* $D(n) = D_n = u_n - u_{n+1}$, *and such that either*

(α) *there is an integer* μ, *positive or zero, for which* $D(x) \geqq D(x')$, *whenever* $x' \geqq x + \mu$; *or*

(β) *there is a constant* k *for which* $D(x)/D(x') < k$, *whenever* $|x - x'| \leqq 1$; *a necessary and sufficient condition for the convergence of the series* $\sum_{n=0}^{\infty} u_n$ *is the convergence of the integral*

$$\int_0^\infty \int_x^\infty D(x)dxdx.$$

Proof. In case (α)

$$D_{n+\mu+1} \leqq \int_n^{n+1} D(x)dx \leqq D_{n-\mu}, \qquad n > \mu. \tag{1}$$

Now

$$\int_n^{n+1}\int_x^m D(x)dxdx = \int_n^{n+1}\left[\overline{\int_x^{n+1} + \int_{n+1}^{n+2} + \int_{n+2}^{n+3} + \cdots + \int_{m-1}^{m}} D(x)dx\right]dx \tag{2}$$

where m is any integer such that $m > n + 1 > \mu + 1$, and where, as elsewhere in this paper, the bar over the signs of integration indicates that all the integrals beneath it have the same integrand. Then, by (1),

$$\begin{aligned}\int_n^{n+1}\int_x^m D(x)dxdx &\leqq D_{n-\mu} + D_{n-\mu+1} + D_{n-\mu+2} + \cdots + D_{m-\mu-1} \\ &= u_{n-\mu} - u_{m-\mu}.\end{aligned} \tag{3}$$

Likewise

$$\begin{aligned}\int_n^{n+1}\int_x^m D(x)dxdx &\geqq D_{n+\mu+2} + D_{n+\mu+3} + \cdots + D_{m+\mu} \\ &= u_{n+\mu+2} - u_{m+\mu+1}.\end{aligned} \tag{4}$$

Therefore, since $\lim_{m=\infty} u_m = 0$,

$$\lim_{m=\infty}\int_n^{n+1}\int_x^m D(x)dxdx$$

exists, and

$$u_{n+\mu+2} \leqq \int_n^{n+1}\int_x^{\infty} D(x)dxdx \leqq u_{n-\mu}. \tag{5}$$

In case (β)

$$\frac{1}{k} D_n < \int_n^{n+1} D(x)dx < kD_n. \tag{6}$$

Then, by (2),

$$\int_n^{n+1}\int_x^{m} D(x)dxdx < k(D_n + D_{n+1} + \cdots + D_{m-1}) = ku_n - ku_m. \tag{7}$$

Likewise

$$\int_n^{n+1}\int_x^{m} D(x)dxdx > \frac{1}{k} u_{n+1} - \frac{1}{k} u_m. \tag{8}$$

Therefore, since $\lim\limits_{m=\infty} u_m = 0$,

$$\lim_{m=\infty}\int_n^{n+1}\int_x^{m} D(x)dxdx$$

exists, and

$$\frac{1}{k} u_{n+1} \leqq \int_n^{n+1}\int_x^{\infty} D(x)dxdx \leqq ku_n. \tag{9}$$

The theorem follows from a comparison of the two series $\sum\limits_{n=0}^{\infty} u_n$ and $\sum\limits_{n=0}^{\infty}\int_n^{n+1}\int_x^{\infty} D(x)dxdx$ by means of (5) and (9).

Example. Test for convergence the following double series

$$\sum_{n=2}^{\infty}\sum_{n=n}^{\infty}\frac{1}{n^2(\log n)^p}\left[1 + \frac{p}{\log n}\right].$$

This can be taken as a double series of which the $(n-1)$-st row is the simple series

$$u_n = \sum_{n=n}^{\infty}\frac{1}{n^2(\log n)^p}\left[1 + \frac{p}{\log n}\right].$$

We see at once that this simple series converges so that u_n exists and $\lim\limits_{n=\infty} u_n = 0$. Write

$$D_n = u_n - u_{n+1} = \frac{1}{n^2(\log n)^p}\left[1 + \frac{p}{\log n}\right].$$

The conditions of the theorem hold, and

$$\int_{x_0}^{\infty}\int_x^{\infty} D(x)dxdx = \int_{x_0}^{\infty}\int_x^{\infty}\frac{1}{x^2(\log x)^p}\left[1 + \frac{p}{\log x}\right]dxdx = \int_{x_0}^{\infty}\frac{dx}{x(\log x)^p}.$$

Therefore the double series converges if $p > 1$, and diverges if $p \leqq 1$.

This method involves essentially the same work as that involved in testing the series by the generalization to double series of the Maclaurin-Cauchy integral test. The present method has the slight advantage that only one variable of integration is used.

4. **The Double-Ratio.** Suppose that we are given the series

$$u_0 + u_1 + u_2 + \cdots.$$

We may call R_n the *double-ratio* of the series if

$$R_n = r_{n+1}/r_n = u_{n+2}\cdot u_n/(u_{n+1})^2.$$

Now suppose that $R(x)$ is a function such that $R(n) = R_n$. Then if $r(x)$ is properly defined in some initial interval the difference equation

$$\log R(x) = \log r(x+1) - \log r(x)$$

defines a function $r(x)$ having the property that $r(n) = r_n$. By the mean-value theorem we are led to the approximation

$$\log R(x) = r'(x)/r(x),$$

or

$$\log r(x) = \int_0^x \log R(x)dx + C.$$

Let us substitute this form in place of $\log r(x)$ in the test of § 2. We get the integral

$$\int_0^\infty e^{\int_0^x \left(\int_0^x \log R(x)dx + C\right)dx} dx.$$

The interesting case is that in which

$$C = -\int_0^\infty \log R(x)dx.$$

The following test is then suggested.

THEOREM VI. *Given the sequence* $\{u_n\}$. *Let* $r_n = u_{n+1}/u_n$ *and* $R_n = r_{n+1}/r_n = u_{n+2}\cdot u_n/(u_{n+1})^2$. *If* $\lim_{n=\infty} r_n = 1$, *and if* $R(x)$ *is a function such that* $R(n) = R_n$, *and such that* $R(x) \geqq R(x')$ *when* $x' > x$, *a necessary and sufficient condition for the convergence of the series* $\sum_{n=0}^{\infty} u_n$ *is the convergence of the integral*

$$\int_0^\infty e^{-\int_0^x \int_x^\infty \log R(x)dxdx} dx.$$

Proof. By definition

$$R_n\cdot R_{n+1}\cdot \cdots \cdot R_m = \frac{1}{r_n}\cdot r_{m+1}, \qquad m > n.$$

Then since $\lim_{m=\infty} r_m = 1$, the infinite product $\prod_{k=n}^{\infty} R_k$ converges to the value $1/r_n$.

Let m and n be integers such that $0 \leqq n < m$. Then since $R(x) \geqq 1$ we have for $n \leqq b \leqq n+1$

$$(1)\quad \int_n^b \int_x^m \log R(x)dxdx \leqq \int_n^b \int_n^m \log R(x)dxdx$$
$$\leqq (b-n)\log(R_n R_{n+1}\cdot \cdots \cdot R_{m-1}).$$

Also

$$(2)\quad \int_n^b \int_x^m \log R(x)dxdx \geqq \int_n^b \int_{n+1}^m \log R(x)dxdx$$
$$\geqq (b-n)\log(R_{n+2}R_{n+3}\cdot \cdots \cdot R_m).$$

Therefore, since $\prod_{k=n}^{\infty} R_k = 1/r_n$,

$$\lim_{m=\infty} \int_n^b \int_x^m \log R(x)dxdx$$

exists, and

$$(3)\quad \int_n^{n+1} \int_x^{\infty} \log R(x)dxdx \leqq \log(1/r_n) = -\log r_n,$$

and

$$(4)\quad \int_n^{n+1} \int_x^{\infty} \log R(x)dxdx \geqq \log(1/r_{n+2}) = -\log r_{n+2}.$$

Now

$$\int_n^{n+1} e^{-\int_0^x \int_x^{\infty} \log R(x)dxdx} dx$$
$$= \int_n^{n+1} e^{-\left[\int_0^1 + \int_1^2 + \int_2^3 + \cdots + \int_{n-1}^{n} + \int_n^x \left(\int_x^{\infty} \log R(x)dx\right)dx\right]} dx.$$

Consequently, by (4),

$$(5)\quad \int_n^{n+1} e^{-\int_0^x \int_x^{\infty} \log R(x)dxdx} dx \leqq \int_n^{n+1} e^{\log(r_2\cdot r_3\cdot \cdots \cdot r_{n+1})}dx$$
$$= r_2\cdot r_3\cdot \cdots \cdot r_{n+1} = \frac{u_{n+2}}{u_2}.$$

And by (3)

$$(6)\quad \int_n^{n+1} e^{-\int_0^x \int_x^{\infty} \log R(x)dxdx} dx \geqq \int_n^{n+1} e^{\log(r_0\cdot r_1\cdot \cdots \cdot r_n)}dx = r_0\cdot r_1\cdot \cdots \cdot r_n = \frac{u_{n+1}}{u_0}.$$

Since the integral given in the theorem does not oscillate, but converges or diverges with the series

$$\sum_{n=0}^{\infty} \int_n^{n+1} e^{-\int_0^x \int_x^{\infty} \log R(x)dxdx} dx,$$

a comparison of this series with the given series $\sum_{n=0}^{\infty} u_n$ by means of (5) and (6) establishes the theorem.

The theorem is easily generalized to include functions $R(x)$ which satisfy conditions similar to the conditions imposed upon $r(x)$ in Theorems II and III. For the sake of brevity it seems better to omit complete statements of these extensions.

Example. Test for convergence the following series

$$1 + e^{-\alpha\left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right)} + e^{-\alpha\left(1 + \frac{2}{2^2} + \frac{2}{3^2} + \frac{2}{4^2} + \cdots\right)} + e^{-\alpha\left(1 + \frac{2}{2^2} + \frac{3}{3^2} + \frac{3}{4^2} + \frac{3}{5^2} + \cdots\right)} + \cdots.$$

Here

$$u_n = e^{-\alpha\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1} + \frac{n-1}{n^2} + \frac{(n-1)}{(n+1)^2} + \frac{(n-1)}{(n+2)^2} + \cdots\right)},$$

$$r_n = e^{-\alpha\left(\frac{1}{n^2} + \frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \cdots\right)},$$

$$R_n = e^{\alpha/n^2}, \qquad \text{and} \qquad R(x) = e^{\alpha/x^2}.$$

Since $R(x)$ decreases monotonically for $\alpha > 0$, and since $\lim_{n=\infty} r_n = 1$, the conditions for the theorem apply. Then

$$-\int_1^x \int_x^\infty \log R(x)\,dx\,dx = -\int_1^x \int_x^\infty \frac{\alpha}{x^2}\,dx = -\int_1^x \frac{\alpha}{x}\,dx = -\alpha \log x.$$

Therefore

$$\int_1^\infty e^{-\int_1^x \int_x^\infty \log R(x)\,dx\,dx}\,dx = \int_1^\infty \frac{dx}{x^\alpha}.$$

Consequently the series converges if $\alpha > 1$, and diverges if $\alpha \leqq 1$.

By the mean-value theorem

$$\frac{\log R(x+h) - \log R(x)}{R(x+h) - R(x)} = \frac{R'(\xi)/R(\xi)}{R'(\xi)} = \frac{1}{R(\xi)}, \tag{7}$$

where $x < \xi < x + h$. Under the conditions of the last theorem, $R(x) \geqq 1$, and $\lim_{x=\infty} R(x) = 1$. Let h become infinite. Then from (7)

$$\log R(x) = [R(x) - 1]\frac{1}{R(\xi)}, \tag{8}$$

where $x < \xi$. Now $1/R(x) \leqq 1$, but if k is any constant less than unity, there is a constant m such that $1/R(x) > k$ when $x > m$. Therefore

$$\log R(x) \leqq R(x) - 1, \tag{9}$$

and

$$\log R(x) > k[R(x) - 1], \tag{10}$$

when $k < 1$ and $x > m$. Then the integral

$$\int_0^\infty e^{-\int_0^x \int_x^\infty \log R(x)dxdx} dx$$

diverges if the integral

$$(A) \qquad \int_0^\infty e^{\int_0^x \int_x^\infty [1-R(x)]dxdx} dx$$

diverges, and it converges if the integral

$$(B) \qquad \int_0^\infty e^{k\int_0^x \int_x^\infty [1-R(x)]dxdx} dx$$

converges. So a sufficient condition for the divergence of the series $u_0 + u_1 + u_2 + \cdots$ is the divergence of the integral (A), and a sufficient condition for the convergence of the series is the convergence of the integral (B), if $0 < k < 1$.

In case neither of these conditions holds, so that the integral (A) converges but the integral (B) diverges, let us assume that there is a constant λ such that

$$(11) \qquad 1 - R(x) > -\frac{\lambda}{x^2}.$$

Under the conditions of convergence and divergence just stated this assumption is commonly true.

We can write

$$(12) \quad \log R(x) = (R(x) - 1) - \tfrac{1}{2}(R(x) - 1)^2 + \tfrac{1}{3}(R(x) - 1)^3 - \cdots.$$

This series converges when x exceeds some fixed value. By assumption (11) we have

$$1 - R(x) > -\frac{\lambda}{x^2}, \qquad (1 - R(x))^2 < \frac{\lambda^2}{x^4}, \qquad (1 - R(x))^3 > -\frac{\lambda^3}{x^6}, \quad \text{etc.}$$

Then since $(1 - R(x)) < 0$,

$$-\log R(x) < (1 - R(x)) + \frac{\lambda^2}{2x^4} + \frac{\lambda^4}{4x^8} + \frac{\lambda^6}{6x^{12}} + \cdots.$$

And if $x > x_0 > \lambda$,

$$-\log R(x) < (1 - R(x)) + \frac{1}{x^4}\left[\frac{\lambda^2}{2} + \frac{\lambda^4}{4x_0^4} + \frac{\lambda^6}{6x_0^8} + \cdots\right] = (1 - R(x)) + \frac{k}{x^4},$$

where k is a positive constant. Consequently

$$-\int_x^\infty \log R(x)dx < \int_x^\infty (1 - R(x))dx + \frac{k}{3x^3}.$$

Integrating again between the limits x_0 and x, we have

$$-\int_{x_0}^{x}\int_{x}^{\infty}\log R(x)dxdx < \int_{x_0}^{x}\int_{x}^{\infty}(1 - R(x))dxdx + \frac{k}{6x_0^2} - \frac{k}{6x^2}$$

$$< \int_{x_0}^{x}\int_{x}^{\infty}(1 - R(x))dxdx + \frac{k}{6x_0^2}.$$

It follows that if the integral

$$\int_{x_0}^{\infty} e^{\int_{x_0}^{x}\int_{x}^{\infty}(1 - R(x))dxdx}dx$$

converges, so also will the integral

$$\int_{x_0}^{\infty} e^{-\int_{x_0}^{x}\int_{x}^{\infty}\log R(x)dxdx}dx.$$

We may therefore state the following theorem.

THEOREM VII. *Given the sequence $\{u_n\}$ and a differentiable function $R(x)$ satisfying the preliminary conditions of Theorem VI. If k is any constant less than unity, a sufficient condition for the convergence of the series $\sum_{n=0}^{\infty} u_n$ is the convergence of the integral*

$$\int_{0}^{\infty} e^{k\int_{0}^{x}\int_{x}^{\infty}[1 - R(x)]dxdx}dx;$$

a necessary condition for the convergence of the series is the convergence of the integral

$$\int_{0}^{\infty} e^{\int_{0}^{x}\int_{x}^{\infty}[1 - R(x)]dxdx}dx,$$

and this condition is also sufficient if

$$(1 - R(x)) > -\frac{\lambda}{x^2}$$

where λ is some constant.

Example. Test for convergence the series

$$\sum_{n=\mu}^{\infty}\frac{1}{(1+\alpha)\left(1+\frac{\alpha}{2^2}\right)^2\left(1+\frac{\alpha}{3^2}\right)^3\cdot\cdots\cdot\left(1+\frac{\alpha}{(n-1)^2}\right)^{n-1} \times\left(1+\frac{\alpha}{n^2}\right)^{n-1}\left(1+\frac{\alpha}{(n+1)^2}\right)^{n-1}\cdots}.$$

For this series

$$r_n = \frac{1}{\left(1+\frac{\alpha}{n^2}\right)\left(1+\frac{\alpha}{(n+1)^2}\right)\left(1+\frac{\alpha}{(n+2)^2}\right)\cdots},$$

and $R_n = 1 + (\alpha/n^2)$, so that $R(x) = 1 + (\alpha/x^2)$. The conditions for the theorem are clearly satisfied.

$$\int_1^x \int_x^\infty (1 - R(x))dxdx = \int_1^x \int_x^\infty -\frac{\alpha}{x^2}dxdx = -\int_1^x \frac{\alpha}{x}dx = -\alpha \log x.$$

Therefore

$$\int_1^\infty e^{\int_1^x \int_x^\infty [1 - R(x)]dxdx} dx = \int_1^\infty \frac{dx}{x^\alpha},$$

so that the given series converges if $\alpha > 1$, and diverges if $\alpha \leqq 1$.

A series for which the double-ratio function is known may be written in the normal form

$$a + b + \frac{b^2}{a}a_0 + \frac{b^3}{a^2}a_0^2 a_1 + \frac{b^4}{a^3}a_0^3 a_1^2 a_2 + \cdots,$$

where a_n is the value $a(n)$ of a known function $a(x)$. It is to such series that the theorems of this section apply.

5. **The Difference of Ratios.** Given a sequence $\{u_n\}$. Let the ratio be $r_n = u_{n+1}/u_n$ and the *difference of ratios* be

$$\rho_n = r_{n+1} - r_n = \frac{u_{n+2}}{u_{n+1}} - \frac{u_{n+1}}{u_n}.$$

If $\rho(x)$ is a function such that $\rho(n) = \rho_n$, and if $r(x)$ is properly defined in some initial interval, the difference equation

$$\rho(x) = r(x + 1) - r(x)$$

defines the function $r(x)$ so that $r(n) = r_n$. Then $\rho(x) = r'(\xi), x < \xi < x + 1$, and we expect that if suitable restrictions are placed upon $\rho(x)$, a relatively small error will be made if, for large values of x, we set $\rho(x) = r'(x)$ or

$$r(x) = \int_0^x \rho(x)dx + C.$$

The integral of § 2 becomes

$$\int_0^\infty e^{\int_0^x \log\left(C + \int_0^x \rho(x)dx\right)dx} dx.$$

In the interesting case we shall have

$$\lim_{x=\infty} \int_0^x \rho(x)dx + C = 1, \qquad \text{or} \qquad C + \int_0^x \rho(x)dx = 1 - \int_x^\infty \rho(x)dx.$$

This suggests the following theorem.

THEOREM VIII. *Given the sequence* $\{u_n\}$. *Let* $r_n = u_{n+1}/u_n$, *and* $\rho_n = r_{n+1} - r_n$. *Suppose that* $r_n < 1$ *and that* $\lim_{n=\infty} r_n = 1$. *Let* $\rho(x)$ *be a*

function such that

(1) $$\rho(n) = \rho_n,$$

(2) $$0 < \rho(x) < 1,$$

(3) $\rho(x)$ *decreases monotonically when* x *increases.*

A necessary and sufficient condition for the convergence of the series $\sum_{n=0}^{\infty} u_n$ *is the convergence of the integral*

$$\int_0^\infty e^{\int_0^x \log\left(1-\int_x^\infty \rho(x)dx\right)dx} dx.$$

Proof.

(1) $$\rho_{n+1} \leqq \int_n^{n+1} \rho(x)dx \leqq \rho_n.$$

If n and m are integers such that $n < m$, and if $n \leqq b \leqq n+1$, we have

(2) $$\begin{aligned} &\int_n^b \log\left(1 - \int_x^m \rho(x)dx\right)dx \\ &\quad \leqq (b-n)\log\left[1 - \overline{\int_{n+1}^{n+2} - \int_{n+2}^{n+3} - \cdots - \int_{m-1}^{m}} \rho(x)dx\right] \\ &\quad \leqq (b-n)\log\left[1 - \rho_{n+2} - \rho_{n+3} - \cdots - \rho_m\right] \\ &\quad = (b-n)\log\left(1 - r_{m+1} + r_{n+2}\right). \end{aligned}$$

Since $r_k < 1$, this logarithm is defined for all values of n and m. Likewise

(3) $$\begin{aligned} &\int_n^b \log\left(1 - \int_x^m \rho(x)dx\right)dx \\ &\quad \geqq (b-n)\log\left[1 - \overline{\int_{n}^{n+1} - \int_{n+1}^{n+2} - \cdots - \int_{m-1}^{m}} \rho(x)dx\right] \\ &\quad \geqq (b-n)\log\left[1 - \rho_{n} - \rho_{n+1} - \cdots - \rho_{m-1}\right] \\ &\quad = (b-n)\log\left(1 - r_{m} + r_{n}\right). \end{aligned}$$

Therefore, since $\lim_{m=\infty} r_m = 1$,

$$\lim_{m=\infty} \int_n^b \log\left(1 - \int_x^m \rho(x)dx\right)dx$$

exists, $n \leqq b \leqq n+1$, and

(4) $$\log r_n \leqq \int_n^{n+1} \log\left(1 - \int_x^\infty \rho(x)dx\right)dx \leqq \log r_{n+2}.$$

Now

$$\int_n^{n+1} e^{\int_0^x \log\left(1-\int_x^\infty \rho(x)dx\right)dx}dx$$
$$= \int_n^{n+1} e^{\overbrace{\int_0^1+\int_1^2+\cdots+\int_{n-1}^n}+\int_n^x \log\left(1-\int_x^\infty \rho(x)dx\right)dx}dx.$$

Consequently, by (4),

$$(5)\quad \int_n^{n+1} e^{\int_0^x \log\left(1-\int_x^\infty \rho(x)dx\right)dx}dx < e^{\log(r_2\cdot r_3\cdot\ldots\cdot r_{n+1})} = r_2\cdot r_3\cdot\ \cdots\ \cdot r_{n+1} = \frac{u_{n+2}}{u_2}.$$

Also

$$(6)\quad \int_n^{n+1} e^{\int_0^x \log\left(1-\int_x^\infty \rho(x)dx\right)dx}dx > e^{\log(r_0\cdot r_1\cdot\ldots\cdot r_n)} = r_0\cdot r_1\cdot\ \cdots\ \cdot r_n = \frac{u_{n+1}}{u_0}.$$

The theorem follows at once from a comparison of the two series $\sum_{n=0}^{\infty} u_n$ and $\sum_{n=0}^{\infty}\left[\int_n^{n+1} e^{\int_0^x \log\left(1-\int_x^\infty \rho(x)dx\right)dx}dx\right]$.

We have the expansion

$$(7)\quad \begin{aligned}&\log\left(1-\int_x^\infty \rho(x)dx\right)\\ &\quad = -\int_x^\infty \rho(x)dx - \frac{1}{2}\left[\int_x^\infty \rho(x)dx\right]^2 - \frac{1}{3}\left[\int_x^\infty \rho(x)dx\right]^3 - \cdots.\end{aligned}$$

Hence $\log\left(1-\int_x^\infty \rho(x)dx\right) < -\int_x^\infty \rho(x)dx$. Therefore if the integral

$$\int_0^\infty e^{-\int_0^x\int_x^\infty \rho(x)dxdx}dx$$

converges, so also will the integral

$$\int_0^\infty e^{\int_0^x \log\left(1-\int_x^\infty \rho(x)dx\right)}dx.$$

In case the first of these integrals diverges, it will often happen that

$$\rho(x) < \frac{\lambda}{x^2},$$

λ being a constant. Let us assume that this condition holds. Then

from (7) we have for $x > x_0 > \lambda$,

$$\begin{aligned} \log\left(1 - \int_x^\infty \rho(x)dx\right) &> -\int_x^\infty \rho(x)dx - \frac{1}{2}\left[\int_x^\infty \frac{\lambda}{x^2}dx\right]^2 \\ &- \frac{1}{3}\left[\int_x^\infty \frac{\lambda}{x^2}dx\right]^3 - \cdots > -\int_x^\infty \rho(x)dx \\ &- \frac{1}{x^2}\left(\frac{\lambda^2}{2} + \frac{\lambda^3}{3x_0} + \frac{\lambda^4}{4x_0^2} - \cdots\right) = -\int_x^\infty \rho(x)dx - \frac{k}{x^2}, \end{aligned} \tag{8}$$

where k is a positive constant. Consequently

$$\begin{aligned} \int_\mu^x \log\left(1 - \int_x^\infty \rho(x)dx\right)dx &> -\int_\mu^x\int_x^\infty \rho(x)dxdx - \int_\mu^x \frac{k}{x^2}dx \\ &= \int_\mu^x\int_x^\infty \rho(x)dxdx + \frac{k}{x} - \frac{k}{\mu} > -\frac{k}{\mu} - \int_\mu^x\int_x^\infty \rho(x)dxdx. \end{aligned} \tag{9}$$

It follows that

$$\int_\mu^x e^{\int_\mu^x \log\left(1 - \int_x^\infty \rho(x)dx\right)dx}dx > C\int_\mu^x e^{-\int_\mu^x\int_x^\infty \rho(x)dxdx}dx.$$

Therefore if the integral

$$\int_\mu^\infty e^{-\int_\mu^x\int_x^\infty \rho(x)dxdx}dx$$

diverges, the integral

$$\int_\mu^\infty e^{\int_\mu^x \log\left(1 - \int_x^\infty \rho(x)dx\right)dx}dx$$

also diverges. We then have the following test.

THEOREM IX. *Given the sequence* $\{u_n\}$. *Let* $r_n = u_{n+1}/u_n < 1$, *and* $\lim_{n=\infty} r_n = 1$. *Let* $\rho(x)$ *be a function such that* $\rho(n) = \rho_n = r_{n+1} - r_n$, $0 < \rho(x) < 1$, *and* $\rho(x)$ *decreases monotonically. The convergence of the integral*

$$\int_0^\infty e^{-\int_0^x\int_x^\infty \rho(x)dxdx}dx$$

is sufficient for the convergence of the series $\sum_{n=0}^\infty u_n$. *Moreover if there is a constant* λ *such that from a certain point on* $\rho(x) < \lambda/x^2$, *the convergence of the integral is necessary for the convergence of the given series.*

Example. Test for convergence the series $\sum_{n=1}^\infty u_n$ where

$$u_n = \left[1 - \sum_{k=\mu}^\infty \frac{\alpha}{k^2}\right]\left[1 - \sum_{k=\mu}^\infty \frac{\alpha}{(k+1)^2}\right]\cdot\cdots\cdot\left[1 - \sum_{k=\mu}^\infty \frac{\alpha}{(k+n-1)^2}\right]$$

μ being so large that $u_n > 0$ for all positive integral values of n. For this series

$$r_n = \frac{u_{n+1}}{u_n} = \left[1 - \sum_{k=\mu}^{\infty} \frac{\alpha}{(k+n)^2}\right].$$

Then $\lim\limits_{n=\infty} r_n = 1$. Also

$$\rho_n = \frac{\alpha}{(\mu+n)^2}, \qquad \rho(x) = \frac{\alpha}{(\mu+x)^2}.$$

The conditions of the theorem are satisfied.

$$-\int_1^x \int_x^\infty \rho(x)dxdx = -\int_1^x \int_x^\infty \frac{\alpha}{(\mu+x)^2} dxdx = \int_1^x \frac{-\alpha}{\mu+x} dx = -\log(\mu+x)^\alpha.$$

Therefore

$$\int_1^\infty e^{-\int_1^x \int_x^\infty \rho(x)dxdx} dx = \int_1^\infty \frac{dx}{(\mu+x)^\alpha},$$

and the given series converges if $\alpha > 1$, and diverges if $\alpha \leqq 1$.

6. **The Ratio of Differences.** Let us call

$$\delta_n = \frac{u_{n+2} - u_{n+1}}{u_{n+1} - u_n}$$

the *ratio of differences* for the series

$$u_0 + u_1 + u_2 + \cdots.$$

Suppose that $\delta(x)$ is a function such that $\delta(n) = \delta_n$. By the same sort of work as that given in the preceding sections, the difference equation in $u(x)$

$$\delta(x) = \frac{u(x+2) - u(x+1)}{u(x+1) - u(x)}$$

gives

$$\delta(x) = \frac{u'(\xi+1)}{u'(\xi)} = e^{u''(\eta)/u'(\eta)}, \qquad x < \xi < \eta < x+2,$$

which suggests as a useful differential equation

$$\log \delta(x) = \frac{u''(x)}{u'(x)}.$$

Then, by the methods already used, the following test is suggested.

THEOREM X. *Given the sequence* $\{u_n\}$ *where* $\lim\limits_{n=\infty} u_n = 0$. *Let*

$$\delta_n = \frac{u_{n+2} - u_{n+1}}{u_{n+1} - u_n},$$

and let $\delta(x)$ be a function having the following properties.

$$(1) \qquad \delta(n) = \delta_n,$$

$$(2) \qquad \delta(x) \leqq 1,$$

$$(3) \qquad \delta(x') \leqq \delta(x), \qquad x' < x.$$

A necessary and sufficient condition for the convergence of the series $\sum_{n=0}^{\infty} u_n$ is the convergence of the integral

$$\int_0^{\infty} \int_x^{\infty} e^{\int_0^x \log \delta(x) dx} dx dx.$$

Proof.

$$(1) \qquad \log \delta_n \leqq \int_n^{n+1} \log \delta(x) dx \leqq \log \delta_{n+1}.$$

$$(2) \qquad \int_n^{n+1} e^{\int_0^x \log \delta(x) dx} dx = \int_n^{n+1} e^{\overbrace{\int_0^1 + \int_1^2 + \cdots + \int_{n-1}^n} \int_n^x \log \delta(x) dx} dx$$
$$\leqq e^{\log(\delta_1 \cdot \delta_2 \cdot \cdots \cdot \delta_n)} = \delta_1 \cdot \delta_2 \cdot \cdots \cdot \delta_n, \qquad n \geqq \mu.$$

Similarly

$$(3) \qquad \int_n^{n+1} e^{\int_0^x \log \delta(x) dx} dx \geqq \delta_0 \cdot \delta_1 \cdot \cdots \cdot \delta_n.$$

Now

$$\int_n^{n+1} \int_x^m e^{\int_0^x \log \delta(x) dx} dx dx$$
$$= \int_n^{n+1} \left[\overbrace{\int_x^{n+1} + \int_{n+1}^{n+2} + \cdots + \int_{m-1}^m} e^{\int_0^x \log \delta(x) dx} \right] dx.$$

Therefore, by (2),

$$(4) \qquad \int_n^{n+1} \int_x^m e^{\int_0^x \log \delta(x) dx} dx dx \leqq [\delta_1 \cdot \delta_2 \cdot \cdots \cdot \delta_n + \delta_1 \cdot \delta_2 \cdot \cdots \cdot \delta_{n+1} + \cdots$$
$$+ \delta_1 \cdot \delta_2 \cdot \cdots \cdot \delta_{m-1}],$$

and by (3),

$$(5) \qquad \int_n^{n+1} \int_x^m e^{\int_0^x \log \delta(x) dx} dx dx \geqq [\delta_0 \cdot \delta_1 \cdot \cdots \cdot \delta_{n+1} + \delta_0 \cdot \delta_1 \cdot \cdots \cdot \delta_{n+2} + \cdots$$
$$+ \delta_0 \cdot \delta_1 \cdot \cdots \cdot \delta_{m-1}].$$

Moreover

$$\delta_0 \cdot \delta_1 \cdot \cdots \cdot \delta_n = \frac{u_{n+2} - u_{n+1}}{u_1 - u_0}.$$

This division by $u_1 - u_0$ is justified by the conditions imposed on $\delta(x)$. It follows that

$$\lim_{m=\infty}[\delta_0\cdot\delta_1\cdot\cdots\cdot\delta_n+\delta_0\cdot\delta_1\cdot\cdots\cdot\delta_{n+1}+\cdots+\delta_0\cdot\delta_1\cdot\cdots\cdot\delta_{m-1}]$$

$$=\lim_{m=\infty}\frac{1}{u_1-u_0}[(u_{n+2}-u_{n+1})+(u_{n+3}-u_{n+2})+\cdots+(u_{m+1}-u_m)]$$

$$=\lim_{m=\infty}\frac{u_{m+1}-u_{n+1}}{u_1-u_0}=\frac{u_{n+1}}{u_0-u_1}.$$

Consequently, by (4) and (5),

$$\int_n^{n+1}\int_x^\infty e^{\int_0^x \log\delta(x)dx}dxdx \leqq \frac{u_{n+1}}{u_1-u_2}, \tag{7}$$

and

$$\int_n^{n+1}\int_x^\infty e^{\int_0^x \log\delta(x)dx}dxdx \geqq \frac{u_{n+2}}{u_0-u_1}. \tag{8}$$

The theorem then follows from a comparison of the two series $\sum_{n=0}^\infty u_n$ and

$$\sum_{n=0}^\infty\int_n^{n+1}\int_x^\infty e^{\int_0^x \log\delta(x)dx}dxdx.$$

Example. Test the double series

$$\sum_{n=2}^\infty\sum_{l=n-1}^\infty e^{-\sum_{k=1}^{l}\frac{p}{k}}.$$

We can write this as $\sum_{n=2}^\infty u_n$ where

$$u_n=\sum_{l=n-1}^\infty e^{-\sum_{k=1}^{l}\frac{p}{k}}.$$

First let us determine under what conditions the series defining u_n converges. We can write

$$u_n=C(1+e^{-\frac{p}{n}}+e^{-\frac{p}{n}-\frac{p}{n+1}}+e^{-\frac{p}{n}-\frac{p}{n+1}-\frac{p}{n+2}}+\cdots)=C+C\sum_{k=0}^\infty v_k,$$

where $v_k=e^{-\frac{p}{n}-\frac{p}{n+1}-\cdots-\frac{p}{n+k}}$. For the series $\sum_{k=0}^\infty v_k$, the ratio is

$$r_k=v_{k+1}/v_k=e^{-\frac{p}{k+n+1}},\qquad\text{and}\qquad r(x)=e^{-\frac{p}{n+x+1}}.$$

We can apply Theorem II. We get

$$\int_0^x e^{\int_\mu^x \log r(x)dx}dx=\int_0^x e^{\int_\mu^x\frac{p}{x+n+1}dx}dx=C_1\int_0^x\frac{dx}{(x+n+1)^p}, \tag{9}$$

and the series $\sum_{k=0}^\infty v_k$ converges, and u_n is defined, if and only if $p>1$. If $p>1$ we see also that $\lim_{n=\infty}u_n=0$, for u_n is the remainder after $(n-2)$

terms of the convergent series that defines u_2. We have

$$\delta_n = \frac{u_{n+2} - u_{n+1}}{u_{n+1} - u_n} = \frac{e^{-\sum_{k=1}^{n} \frac{p}{k}}}{e^{-\sum_{k=1}^{n-1} \frac{p}{k}}} = e^{-\frac{p}{n}},$$

$$\delta(x) = e^{-\frac{p}{x}}.$$

We may apply Theorem X. As in (9), we have

$$(10) \qquad \int_x^\infty e^{\int_0^x \log \delta(x) dx} dx = C \int_x^\infty \frac{dx}{x^p} = \frac{C}{(p-1)x^{p-1}},$$

$$(11) \qquad \int_0^\infty \int_x^\infty e^{\int_0^x \log \delta(x) dx} dx dx = \frac{C}{p-1} \int_0^\infty \frac{dx}{x^{p-1}}.$$

Therefore the given series converges if $p > 2$, and diverges if $p \leqq 2$. This same series is conveniently tested by means of a generalization to multiple series of the test of Theorem II.

7. **Other Tests.** The following test is immediately suggested and established by the methods already used.

THEOREM XI. *Given the sequence* $\{u_n\}$. *Let* $D_n = u_n - u_{n+1}$, *and* $t_n = D_n - D_{n+1}$. *If* $\lim_{n=\infty} u_n = 0$, *and* $t(x)$ *be a positive function such that* $t(n) = t_n$, *and* $t(x') \geqq t(x)$, *when* $x > x'$, *then a necessary and sufficient condition for the convergence of the series* $\sum_{n=0}^{\infty} u_n$ *is the convergence of the integral*

$$\int_0^\infty \int_x^\infty \int_x^\infty t(x) dx dx dx.$$

The difference equation of the fourth order for $S(x)$ that leads to the following test, as well as the proof of the test are readily suggested by the methods used elsewhere in this paper.

THEOREM XII. *Given the sequence* $\{u_n\}$. *Let* $r_n = u_{n+1}/u_n$, $R_n = r_{n+1}/r_n$, *and*

$$\rho_n = R_{n+1}/R_n = \frac{u_{n+3} \cdot u_{n+1}^3}{u_n \cdot u_{n+2}^3}.$$

If $\lim_{n=\infty} r_n = 1$, *and* $\rho(x)$ *be a positive function such that* $\rho(n) = \rho_n$, *and* $\rho(x) \leqq 1$, *and* $\rho(x') \geqq \rho(x)$ *when* $x' > x$, *then a necessary and sufficient condition for the convergence of the series* $\sum_{n=0}^{\infty} u_n$ *is the convergence of the integral*

$$\int_0^\infty e^{\int_0^x \int_x^\infty \int_x^\infty \log \rho(x) dx dx dx} dx.$$

8. Conclusion. From the examples given, the general method is revealed by which one can obtain an infinite sequence of tests, of which those given above are the simplest and most useful.

Tests for multiple series are readily suggested by analogy with the foregoing tests for simple series. A test for multiple series analogous to the test of Theorem II is given in the author's paper to which reference has already been made. As is shown by some of the examples of this paper some of the foregoing tests can be used directly to examine certain types of multiple series.

The theorems of this paper have been stated for series of constant terms. Many of the tests can readily be extended to series of functions, not only to test such series for convergence, but also to determine whether the convergence is uniform, uniform convergence of an integral implying uniform convergence of the corresponding series.

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CALCULATION OF THE COMPLEX ZEROS OF THE FUNCTION $P(z)$ COMPLEMENTARY TO THE INCOMPLETE GAMMA FUNCTION.*

By Philip Franklin.

The gamma function of the complex variable may be written as the sum of an integral function and the function

$$P(z) = \sum_{\nu=0}^{\infty} \frac{(-1)^\nu}{\nu!} \frac{1}{z+\nu}$$

which has the same poles and the same residues as $\Gamma(z)$ itself. The fact that $P(z) = 0$ has exactly four complex roots was first stated, although with an insufficient proof, by L. Bourguet.† The method of computing these four complex roots used here is based on an existence proof given recently by T. H. Gronwall.‡ This proof consists in first showing that the auxiliary rational function

$$P_n(z) = \sum_{\nu=0}^{2n-1} \frac{(-1)^\nu}{\nu!} \frac{1}{z+\nu} \qquad (n > 2)$$

has all its real roots lying one in each of the $2n - 3$ intervals:

$$-2m - 1 - \frac{6}{(2m)!} \leqq z \leqq -2m - 1 - \frac{1}{(2m)!} \qquad (m = 2, 3, \cdots, n)$$

$$-2m - 2 + \frac{1}{(2m+1)!} \leqq z \leqq -2m - 2 + \frac{6}{(2m+1)!} \qquad (m=2, 3, \cdots, n-1),$$

and hence that $P_n(z)$ has four complex roots and from this proceeding to $P(z)$ itself by applying Rouché's theorem. The method of computation consists in locating the complex zeros of $P_{10}(z)$ to three decimal places and then, using these as initial values, finding the roots of $P(z) = 0$ by a direct application of Newton's method of approximation.

Starting with the mean values of the above-mentioned intervals for $n = 10$, and applying Newton's method of approximation to

$$P_{10}(z) = \sum_{\nu=0}^{21} \frac{(-1)^\nu}{\nu!} \frac{1}{z+\nu} \tag{1}$$

* Presented at the meeting of the American Mathematical Society, at New York City, April 26, 1919.

† L. Bourguet, Sur la théorie des intégrales Euleriennes, Comptes rendus, vol. 96 (1883), pp. 1487–1490.

‡ T. H. Gronwall, Sur les zéros des fonctions $P(z)$ et $Q(z)$ associées à la fonction gamma, Annales de l'Ecole Normale, Ser. III, vol. 33 (1916), pp. 381–393.

we find that the real roots are

$$p_5 = -5.11907; \quad p_6 = -5.97158; \quad p_7 = -7.00432;$$

$$p_8 = -7.99940; \quad p_9 = -9.00008; \quad p_{10} = -9.99999;$$

$$p_m = -m \qquad (m = 11, 12, \cdots 21),$$

where, as in the rest of this computation, quantities which do not affect the fifth decimal place are neglected. Hence $P_{10}(z)$ which may be written as

$$\frac{a_0z^{21} + a_1z^{20} + \cdots + a_{20}z + a_{21}}{z(z+1)(z+2)\cdots(z+21)}$$

is equal to

$$(2) \qquad \frac{b_0z^{10} + b_1z^9 + \cdots + b_9z + b_{10}}{z(z+1)(z+2)\cdots(z+10)}$$

to the desired degree of accuracy.

If $P_{10}(z)$ be expanded in powers of $1/z$, the coefficient of $1/z^r$ is

$$\sum_{\nu=1}^{21} \frac{(-1)^{\nu+r}\nu^r}{\nu!},$$

and since this is sensibly equal to

$$\sum_{\nu=1}^{\infty} \frac{(-1)^{\nu+r}\nu^r}{\nu!} \qquad (r < 10),$$

$$(3) \qquad \frac{1}{e}\left(\frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^4} + \frac{1}{z^5} + \cdots\right)$$

is obtained as a second approximate form for $P_{10}(z)$. A comparison of (3) with (2) gives $b_0 = 1/e$, $b_1 = 56/e$, $b_2 = 1375/e$.

The expansion of $P_{10}(z)$ in the form

$$(4) \qquad \frac{1}{z} + A_1 + A_2z + A_3z^2 + \cdots$$

(where $A_1 = \Sigma_{\nu=1}^{21}[(-1)^\nu/\nu!\nu] = .7965996$) when compared with (2) determines the values $b_9 = 7{,}737{,}939$, $b_{10} = 3{,}628{,}800$.

The numerator of (2) (with the factor $1/e$ removed) is therefore equal to

$$(5) \qquad z^{10} + 56z^9 + 1375z^8 + \cdots + 21{,}033{,}900z + 9{,}864{,}102.$$

This expression, set equal to zero, has for roots the four complex roots of $P_{10}(z) = 0$, together with $p_5, p_6, \cdots p_{10}$ found above. The product $\Pi_{m=5}^{10}(z - p_m)$ is found to be

$$(6) \qquad z^6 + 45.0944z^5 + 355.917z^4 + \cdots + 129{,}749z + 154{,}152.$$

Dividing (5) by (6), we obtain the equation

$$z^4 + 10.906z^3 + 44.425z^2 + 82.589z + 63.989 = 0. \tag{7}$$

The determination of the fourth and fifth terms of (7) from the last two terms of (5) and (6) is preferable to the use of the first five terms of (5) and (6), as it involves fewer operations and consequently gives these terms more accurately than the alternative method.

The roots of (7), obtained by the use of Ferrari's formulæ, are, to three decimal places:

$$p_1 = -1.726 + 1.238i, \qquad p_2 = -1.726 - 1.238i,$$

$$p_3 = -3.727 + .544i, \qquad p_4 = -3.727 - .544i.$$

Starting with these values, and applying Newton's method of approximation to $P(z)$ itself, we find that the four complex roots of $P(z)$ are

$$z_1 = -1.7262976 + 1.2380921i, \qquad z_2 = -1.7262976 - 1.2380921i,$$

$$z_3 = -3.7264730 + .5406746i, \qquad z_4 = -3.7264730 - .5406746i.$$

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we find that the real roots are

$$p_5 = -5.11907; \qquad p_6 = -5.97158; \qquad p_7 = -7.00432;$$

$$p_8 = -7.99940; \qquad p_9 = -9.00008; \qquad p_{10} = -9.99999;$$

$$p_m = -m \qquad (m = 11, 12, \cdots 21),$$

where, as in the rest of this computation, quantities which do not affect the fifth decimal place are neglected. Hence $P_{10}(z)$ which may be written as

$$\frac{a_0 z^{21} + a_1 z^{20} + \cdots + a_{20} z + a_{21}}{z(z+1)(z+2)\cdots(z+21)}$$

is equal to

$$\frac{b_0 z^{10} + b_1 z^9 + \cdots + b_9 z + b_{10}}{z(z+1)(z+2)\cdots(z+10)} \tag{2}$$

to the desired degree of accuracy.

If $P_{10}(z)$ be expanded in powers of $1/z$, the coefficient of $1/z^r$ is

$$\sum_{\nu=1}^{21} \frac{(-1)^{\nu+r}\nu^r}{\nu!},$$

and since this is sensibly equal to

$$\sum_{\nu=1}^{\infty} \frac{(-1)^{\nu+r}\nu^r}{\nu!} \qquad (r < 10),$$

$$\frac{1}{e}\left(\frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^4} + \frac{1}{z^5} + \cdots\right) \tag{3}$$

is obtained as a second approximate form for $P_{10}(z)$. A comparison of (3) with (2) gives $b_0 = 1/e$, $b_1 = 56/e$, $b_2 = 1375/e$.

The expansion of $P_{10}(z)$ in the form

$$\frac{1}{z} + A_1 + A_2 z + A_3 z^2 + \cdots \tag{4}$$

(where $A_1 = \Sigma_{\nu=1}^{21}[(-1)^\nu/\nu!\nu] = .7965996$) when compared with (2) determines the values $b_9 = 7{,}737{,}939$, $b_{10} = 3{,}628{,}800$.

The numerator of (2) (with the factor $1/e$ removed) is therefore equal to

$$z^{10} + 56z^9 + 1375z^8 + \cdots + 21{,}033{,}900z + 9{,}864{,}102. \tag{5}$$

This expression, set equal to zero, has for roots the four complex roots of $P_{10}(z) = 0$, together with $p_5, p_6, \cdots p_{10}$ found above. The product $\Pi_{m=5}^{10}(z - p_m)$ is found to be

$$z^6 + 45.0944z^5 + 355.917z^4 + \cdots + 129{,}749z + 154{,}152. \tag{6}$$

Dividing (5) by (6), we obtain the equation

$$(7) \qquad z^4 + 10.906z^3 + 44.425z^2 + 82.589z + 63.989 = 0.$$

The determination of the fourth and fifth terms of (7) from the last two terms of (5) and (6) is preferable to the use of the first five terms of (5) and (6), as it involves fewer operations and consequently gives these terms more accurately than the alternative method.

The roots of (7), obtained by the use of Ferrari's formulæ, are, to three decimal places:

$$p_1 = -1.726 + 1.238i, \qquad p_2 = -1.726 - 1.238i,$$

$$p_3 = -3.727 + .544i, \qquad p_4 = -3.727 - .544i.$$

Starting with these values, and applying Newton's method of approximation to $P(z)$ itself, we find that the four complex roots of $P(z)$ are

$$z_1 = -1.7262976 + 1.2380921i, \qquad z_2 = -1.7262976 - 1.2380921i,$$

$$z_3 = -3.7264730 + .5406746i, \qquad z_4 = -3.7264730 - .5406746i.$$

ORDNANCE DEPARTMENT,
ABERDEEN PROVING GROUNDS.

TOTAL DIFFERENTIABILITY.

By E. J. Townsend.

Suppose we have given a single-valued function $z = f(x, y)$ of two real variables, defined for a region R given by the inequalities

$$a < x < b, \qquad c < y < d.$$

Thomae* seems to have been the first to point out that the mere existence of the partial derivatives $f_x'(x_0, y_0)$, $f_y'(x_0, y_0)$ is not sufficient for the total differentiability of the given function at the point (x_0, y_0).

More recently several writers† have formulated a more precise definition of a total differentiable. These definitions are, however, equivalent and may be stated as follows:

For convenience, denote by $\Delta(x, y)$ the distance $\sqrt{(\Delta x)^2 + (\Delta y)^2}$ between any given point (x_0, y_0) of R and any other such point $(x_0 + \Delta x, y_0 + \Delta y)$. The function $f(x, y)$ is then said to be totally differentiable at (x_0, y_0) if there exist two constants A, B such that

$$(1) \qquad \underset{\substack{\Delta x \doteq 0 \\ \Delta y \doteq 0}}{\mathrm{L}} \frac{f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0) - A\Delta x - B\Delta y}{\Delta(x, y)} = 0.$$

As Fréchet points out, one might make use of $|\Delta x| + |\Delta y|$ instead of $\Delta(x, y)$. It follows from this definition that if the given function is totally differentiable, the partial derivatives $f_x'(x_0, y_0)$, $f_y'(x_0, y_0)$ both exist and are finite, A being nothing else than $f_x'(x_0, y_0)$ and B, $f_y'(x_0, y_0)$.

Such a definition meets adequately the needs of analysis and has been adopted in one form or another in recent texts.‡

We shall examine some of the consequences of such a definition, particularly the conditions that must be placed upon the partial derivatives f_x', f_y' in order that the given function may be said to be totally differentiable. As an aid to the geometrical interpretation of the results of such a study, it may be pointed out that a necessary and sufficient condition for total differentiability for $x = x_0$, $y = y_0$ is that the surface $z = f(x, y)$ shall

* See Theorie der Bestimmten Integrale (1875), p. 36.

† See Stotz, Differential- und Integral-rechnung (1893), p. 131; W. H. Young, Fundamental Theorems of Differential Calculus (1910), p. 21; Fréchet, Nouvelles Annalles de Math. (1912), vol. 71, p. 389.

‡ See Pierpont, Theory of Functions of Real Variables, vol. I, p. 269; Kowalewski, Komplexen Veränderlichen, p. 186; de la Vallée Poussin, Cours d'Analyse Infinitesimale, 3d ed., p. 140.

have a tangent plane at the point (x_0, y_0, z_0), which is not parallel to the z-axis.*

Continuity of $f(x, y)$ as regards the two variables taken together is a consequence of total differentiability, but as with functions of a single variable a given function may be continuous throughout a given region but not be totally differentiable at any point of the region. Total differentiability depends upon the existence of the partial derivatives f_x', f_y', and the character of their continuity.

If f_x', f_y' both exist and one is continuous in x and y together, then it follows that $f(x, y)$ is totally differentiable.† It is well known that a function of two variables which is continuous in each variable throughout a region is also continuous in both together at a set of points everywhere dense in that region. It follows then that if f_x' and f_y' exist and one is continuous in x and in y, then $f(x, y)$ is totally differentiable at a set of points everywhere dense. The question naturally arises whether under the foregoing conditions the given function is not totally differentiable at every point. This is not the case, however, as the following illustration shows.

Ex. 1. Given the function

$$f(x, y) = \frac{x^2y}{x^4 + y^2},$$

where $0 < n \leq 1$, $0 < y \leq 1$, and $f(0, 0) = 0$. This function is not continuous in x and y together at the origin; for, we obtain different limiting values by taking different approaches to that point. As continuity of $f(x, y)$ in x and y together is a necessary condition for total differentiability, it follows that the given function is not totally differentiable for $x = 0$, $y = 0$.

For $x \neq 0$, $y \neq 0$, we have

$$f_x' = 2xy \cdot \frac{y^2 - x^4}{(y^2 + x^4)^2}.$$

For $(x = 0, y = 0)$, for $(x \neq 0, y = 0)$, and for $(x = 0, y \neq 0)$, we have $f_x' = 0$. For $(x \neq 0, y \neq 0)$, we have

$$f_y' = x^2 \frac{x^4 - y^2}{(x^4 + y^2)^2};$$

while for $(x = 0, y = 0)$, $(x = 0, y \neq 0)$, we have $f_y' = 0$. For $(x \neq 0, y = 0)$, we obtain $f_y' = 1/x^2$.

It follows then that the given function is not totally differentiable at

* Cf. Fréchet, Nouvelles Annales de Math., vol. 71, p. 436.

† Stolz, Differential- und Integral-rechnung, vol. I, p. 134.

the origin, yet in every neighborhood of that point including the origin itself, the two partial derivatives f_x', f_y' exist and one of them, namely f_x', is continuous in x alone and in y alone. The partial derivative f_y', is continuous everywhere with respect to y. It is also continuous with respect to x, except for $x = 0$, $y = 0$.

As already pointed out, if one of the partial derivatives, f_x', f_y', is continuous in x and in y, then the points at which $f(x, y)$ is totally differentiable forms a set everywhere dense in the given region. However, the points at which the given function $f(x, y)$ is not totally differentiable may also form a set of points everywhere dense in the same region, as the following example shows.

Ex. 2. Consider the function

$$F(s, t) = \sum_{n=1}^{\infty} \frac{1}{n!} \varphi_n(s, t),$$

where $\varphi_n(s, t)$ is formed from the function considered in *Ex.* 1 by replacing x by $\sin^2 n\pi s$ and y by $\sin^2 n\pi t$; that is

$$\varphi_n(s, t) = \frac{\sin^4 n\pi s \cdot \sin^2 n\pi t}{\sin^8 n\pi s + \sin^4 n\pi t}.$$

The function given in *Ex.* 1 takes the value one-half for $y = x^2$ but for all other values of x and y in the given region it is less than one-half. The amount of the (x, y)-discontinuity at the origin is one-half. Since the sine is never greater than unity, we have

$$\sum_{n=1}^{\infty} \frac{1}{n!} \varphi_n(s, t) < \sum_{n=1}^{\infty} \frac{1}{n!}.$$

As the latter series converges, the series $\Sigma(1/n!)\varphi_n(s, t)$ converges uniformly as a function of the two variables (s, t) and hence as a function of either variable separately. As we shall see, the function $F(s, t)$ is not continuous in s and t together throughout the region of definition, because each term of the series is discontinuous at certain points in these two variables. However, $\varphi_n(s, t)$ is continuous in s alone and in t alone; and because of the uniform convergence of the above series, $F(s, t)$ is continuous in each variable separately. Consequently, the points at which it is continuous in both variables must form a set everywhere dense.

The points at which $F(s, t)$ has a discontinuity in s and t together also form a set everywhere dense; being those points where both s and t have rational values, as we shall now show. We have

$$\begin{aligned} F(s, t) = {} & \frac{\sin^4 \pi s \cdot \sin^2 \pi t}{\sin^2 \pi s + \sin^4 \pi t} + \frac{1}{2!} \cdot \frac{\sin^4 2\pi s \cdot \sin^2 2\pi t}{\sin^8 2\pi s + \sin^4 2\pi t} + \cdots \\ & + \frac{1}{n!} \frac{\sin^4 n\pi s \cdot \sin^2 n\pi t}{\sin^8 n\pi s + \sin^4 n\pi t} + \cdots. \end{aligned} \tag{2}$$

The first term of this series has a discontinuity in s and t at

$$(0, 0),\ (0, 1),\ (1, 0),\ (1, 1);$$

the second term at the points

$$(0, 0),\ (0, \tfrac{1}{2}),\ (0, 1),\ (\tfrac{1}{2}, 0),\ (\tfrac{1}{2}, \tfrac{1}{2}),\ (\tfrac{1}{2}, 1),\ (1, 0),\ (1, \tfrac{1}{2}),\ (1, 1).$$

In the general term we have such a singularity at points for which s and t have any combination of the following values:

$$s = 0,\ \frac{1}{n},\ \frac{2}{n},\ \frac{3}{n},\ \cdots,\ \frac{n-1}{n},\ 1,$$

$$t = 0,\ \frac{1}{n},\ \frac{2}{n},\ \frac{3}{n},\ \cdots,\ \frac{n-1}{n},\ 1.$$

It follows that any point whose coördinates (s, t) are both rational numbers is a point of discontinuity of some term in the series defining $F(s, t)$. It will be observed that none of the terms of the series given in (2) are ever negative, whatever values may be assigned to s and t. The points of discontinuity of any term are likewise points of discontinuity of all subsequent terms but not necessarily of previous terms. Moreover, the discontinuities at a given point can not be combined so as to cancel each other. For, if a point first appears as a discontinuity, say in the kth term, the sum of the discontinuities of subsequent terms at that point can not equal in amount the discontinuity of the kth term. For example, consider the discontinuity which appears at the rational point $(x = \frac{1}{2}, y = \frac{1}{3})$. This point will appear as a discontinuity for the first time in the third term. The amount of the discontinuity in this term is $\frac{1}{2} \cdot \frac{1}{3} = \frac{1}{2} \cdot \frac{1}{6}$. The amount of the disconinuity of subsequent terms at this point can not exceed

$$\frac{1}{4!} \cdot \frac{1}{2} + \frac{1}{5!} \cdot \frac{1}{2} + \cdots < \frac{1}{2} \cdot \frac{1}{4!}\left(1 + \frac{1}{5} + \frac{1}{5^2} + \frac{1}{5^3} + \cdots\right) = \frac{1}{2} \cdot \frac{1}{4!} \cdot \frac{5}{4} < \frac{1}{2} \cdot \frac{1}{6}.$$

Consequently, all points which appear as points of discontinuity of any term are also points of discontinuity of $F(s, t)$, and hence any point whose coördinates are both rational numbers is such a point. A function must be continuous in the two variables together in order to be totally differentiable. It follows then that $F(s, t)$ as defined is not totally differentiable at any point where the two coördinates are both rational numbers. Such a set of points is everywhere dense in the given region.

We shall now show that the points at which $F(s, t)$ is totally differentiable are also everywhere dense. To do this we proceed as follows. As we have seen, f_x' is continuous in x in the closed interval $0 \leqq x \leqq 1$;

hence for any constant value of y, say $y = y_0$, the numerical values of $f_x'(x, y_0)$ have a finite upper bound. This upper bound, which we shall denote by M_0, may change with the choice of y_0, but the essential thing is that for each value of y_0 it is a constant. We have then, since $\sin n\pi s$ and $\cos n\pi s$ are numerically less than one,

$$\left|\frac{\partial \varphi_n}{\partial s}\right| = \left|\frac{\partial f}{\partial x}\cdot\frac{dx}{ds}\right| \leqq 2\pi n M_0 \,|\sin n\pi s \cdot \cos n\pi s\,| \leqq 2\pi n \cdot M_0.$$

Hence we have for a constant value of t

$$\sum_{n=1}^{\infty}\left|\frac{1}{n!}\,\frac{\partial \varphi_n}{\partial s}\right| \leqq \sum_{n=1}^{\infty}\frac{2\pi M_0}{(n-1)!}.$$

As this last series converges, it follows that the series $\Sigma(1/n!)/(\partial\varphi_n/\partial s)$ converges uniformly, and we may write

$$\frac{\partial F}{\partial s} = \sum_{n=1}^{\infty}\frac{1}{n!}\,\frac{\partial \varphi_n}{\partial s}.$$

Moreover, since $\partial\varphi_n/\partial s$ is continuous in s, it follows that $\partial F/\partial s$ is defined by a uniformly convergent series of continuous functions and is therefore itself continuous in s.

By a similar method it may be shown that $\partial F/\partial s$ is continuous in t; and since this derivative is also continuous in s, it follows that there exists a set of points everywhere dense where it is continuous in s and t together.

We shall now consider the existence of the partial derivative $\partial F/\partial t$. From *Ex.* 1, it follows that f_y' is continuous in y in the closed interval $0 \leqq y \leqq 1$. Then, for any constant value of x it is bounded, and hence we may write

$$\left|\frac{\partial \varphi_n}{\partial t}\right| = \left|\frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial t}\right| \leqq 2n\pi M_1 \,|\sin n\pi t \cdot \cos n\pi t\,| \leqq 2n\pi M_1,$$

where M_1 is a constant for each previously selected value of x. Consequently, we have

$$\sum_{n=1}^{\infty}\frac{1}{n!}\,\frac{\partial \varphi_n}{\partial t} \leqq \sum_{n=1}^{\infty}\frac{2\pi M_1}{(n-1)!}.$$

Since this last series converges, it follows that $\Sigma(1/n!)(\partial\varphi_n/\partial t)$ converges uniformly, and hence we may write

$$\frac{\partial F}{\partial t} = \sum_{n=1}^{\infty}\frac{1}{n!}\,\frac{\partial \varphi_n}{\partial t}.$$

Therefore, the partial derivatives $\partial F/\partial s$, $\partial F/\partial t$ both exist at every point in the given region and $\partial F/\partial s$ is continuous in s and t together at a set of

points everywhere dense. It follows that $F(s, t)$ is totally differentiable at each of these points.

We have then a function which is totally differentiable at a set of points everywhere dense, and at the same time there exists a set of points likewise everywhere dense at which this function is not totally differentiable, in spite of the fact that both of the first partial derivatives exist at every point and one of them is everywhere continuous in each variable separately. Geometrically, the corresponding surface has a tangent plane at a set of points everywhere dense and there exists simultaneously another set of points at which there is no tangent plane. It can be shown, moreover, that the points at which the tangent planes exist, that is, the points at which the given function is totally differentiable, must form a set having the cardinal number of the continuum. It will be observed that $\partial F/\partial t$ is not continuous with respect to s. By the method used in discussing the continuity of $F(s, t)$, it can be shown that $\partial F/\partial t$ is discontinuous in s at a set of points everywhere dense, namely at the points where s and t are rational numbers.

It has already been pointed out that the mere existence of the partial derivatives f_x', f_y' at a given point is not a sufficient condition for the total differentiability of $f(x, y)$ at that point. The question naturally arises whether it is possible to find a function having these partial derivatives at all points of a given region and yet not be totally differentiable at any point of that region. It can be readily shown that this cannot be the case. For it is known* that if f_x', f_y' exist, they can be at most pointwise discontinuous in the two variables x and y. The points of continuity in x and y together are then everywhere dense and these points are points of total differentiability of the given functions.

If we assume, in addition to the existence of the partial derivatives f_x', f_y', the continuity of these derivatives with respect to x and with respect to y, then we have the following theorem.

THEOREM. *Given a function $f(x, y)$ whose partial derivatives f_x', f_y' are continuous in x and in y and bounded as to x and y taken together in a closed region R. Then $f(x, y)$ is totally differentiable at all points of R.*

From the existence of the partial derivatives f_x', f_y' it follows that $f(x, y)$ is continuous in x and in y. Let (x_0, y_0) be any point in the given region R. Since $f(x, y)$ is continuous with respect to y, we have for each value of x in any closed interval lying wholly in R

$$\mathop{\mathrm{L}}_{y \doteq y_0} f(x, y) = f(x, y_0).$$

Moreover, f_x' is continuous in x in any such interval for $y \neq y_0$. By

* See Baire, Annali di Mat., Series III, vol. 3 (1899), p. 108.

hypothesis, f_x' is also bounded, when considered as a function of the two variables (x, y) together. It follows that for any closed interval taken on $y = y_0$, and lying in R the given function $f(x, y)$ converges uniformly* to the function $f(x, y_0)$. Consequently we have†

$$\mathop{\mathrm{L}}_{\substack{x = x_0 \\ y = y_0}} f(x, y) = f(x_0, y_0);$$

or what is the same thing, for an arbitrarily small positive number η, there exists a $\lambda > 0$ such that

$$(3) \qquad |f(x, y) - f(x_0, y_0)| < \frac{\eta}{2}, \qquad |x - x_0| < \lambda, \qquad |y - y_0| < \lambda.$$

Hence, for a given value of η, however small it may be chosen, $\lambda(x, y)$ is defined for each point (x, y) in R. The function $f(x, y)$ is therefore continuous in x and y together in the closed region R, and hence it is uniformly continuous in R and $\lambda(x, y)$ has a lower limit λ_0 greater than zero. There exists then about each point of R as a center a square S_0 whose sides are of length $2\lambda_0$ such that the oscillation of $f(x, y)$ within the square is less than η. We may regard the point (x_0, y_0) the center of such a square.

Since the partial derivative f_x' exists at the point (x_0, y_0), we have for all values of $x_0 + \Delta x$ within an interval $[x_0 - \delta_1(x_0, y_0), x_0 + \delta_1(x_0, y_0)]$

$$(4) \qquad \left| \frac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{\Delta x} - f_x'(x_0, y_0) \right| < \eta.$$

It follows that

$$(5) \qquad |f(x_0 + \Delta x, y_0) - f(x_0, y_0) - \Delta x f_x'(x_0, y_0)| < \eta\, |\Delta x|.$$

Since the oscillation of $f(x, y)$ in S_0 is less than η, we have for $|\Delta x| < \lambda_0$, $|\Delta y| < \lambda_0$

$$(6) \qquad |f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0 + \Delta x, y_0)| < \eta,$$

$$(7) \qquad |f(x_0, y_0) - f(x_0, y_0 + \Delta y)| < \eta.$$

Combining (5), (6), and (7), we get

* If $f(x, y)$ is continuous in y for each value of x in an interval $\alpha \leqq x \leqq \beta$, and if f_x' exists and

$$|f_x'(x, y)| \leqq G, \qquad \alpha \leqq x \leqq \beta, \qquad y_0 - \delta \leqq y \leqq y_0 + \delta,$$

where G is a finite number, then $f(x, y)$ converges uniformly to the function $f(x, y_0)$. Cf. Townsend, *Begriff u. Anwendung des Doppellimes* (Göttingen Dissertation), p. 34.

† The necessary and sufficient condition that $f(x, y)$ shall converge uniformly to the boundary function $f(x, y_0)$ in the interval (α, β) is that the double simultaneous limit

$$\mathop{\mathrm{L}}_{\substack{x = x_0 \\ y = y_0}} f(x, y) = f(x_0, y_0)$$

for each point in the closed interval $\alpha \leqq x \leqq \beta$. See Townsend, *Ibid.*, p. 39.

(8) $\quad |f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0 + \Delta y) - \Delta x f_x'(x_0, y_0)| < \eta(2 + |\Delta x|)$.

From the existence of the partial derivative $f_y'(x_0, y_0)$, we have for some interval $[y_0 - \delta_2(x_0, y_0), y_0 + \delta_2(x_0, y_0)]$

$$\left| \frac{f(x_0, y_0 + \Delta y) - f(x_0, y_0)}{\Delta y} - f_y'(x_0, y_0) \right| < \eta,$$

whence

(9) $$|f(x_0, y_0 + \Delta y) - f(x_0, y_0) - \Delta y f_y'(x_0, y_0)| < \eta |\Delta y|.$$

By adding and subtracting $f(x_0, y_0 + \Delta y)$ to the numerator of the first member of the following equation, we have the identity

$$\frac{f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0)}{\Delta(x, y)}$$
$$= \frac{f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0 + \Delta y)}{\Delta(x, y)} + \frac{f(x_0, y_0 + \Delta y) - f(x_0, y_0)}{\Delta(x, y)}.$$

Transposing all the terms to the first member of the equation, we have upon making use of the relations given in (8) and (9)

$$\left| \frac{f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0) - \Delta x f_x'(x_0, y_0) - \Delta y f_y'(x_0, y_0)}{\Delta(x, y)} \right|$$
$$< \eta(2 + |\Delta x| + |\Delta y|),$$

which holds for all values of $(\Delta x, \Delta y)$ numerically less than δ_0 where δ_0 denotes the smallest of the three numbers λ_0, δ_1, δ_2. The second member of this inequality is arbitrarily small, since η is an arbitrarily small number. Hence the limit given in (7) exists, and the given function is totally differentiable at (x_0, y_0).

That the foregoing theorem gives a sufficient but not a necessary condition for total differentiability is at once evident from the following illustrative example.

Ex. 3. Given $f(x, y) = (x^2 + y^2) \sin 1/(x + y)$ for $x \neq 0$, $y \neq 0$ and $x \neq -y$, and let $f(x, y) = 0$ for $(x = 0, y = 0)$ and for $x = -y$.

This function has at $(0, 0)$ the partial derivatives $f_x'(0, 0)$, $f_y'(0, 0)$; for, we have for $x = 0$, $y = 0$,

$$f_x'(0, 0) = \underset{\Delta x \doteq 0}{\mathrm{L}} \frac{f(\Delta x, 0) - f(0, 0)}{\Delta x} = \underset{\Delta x \doteq 0}{\mathrm{L}} \Delta x \sin \frac{1}{\Delta x} = 0,$$

$$f_y'(0, 0) = \underset{\Delta y \doteq 0}{\mathrm{L}} \frac{f(0, \Delta y) - f(0, 0)}{\Delta y} = \underset{\Delta y \doteq 0}{\mathrm{L}} \Delta y \sin \frac{1}{\Delta y} = 0.$$

For $x = 0$, $y \neq 0$, we obtain $f_x'(0, y) = -\cos 1/y$. Finally, for $x \neq 0$, $y \neq 0$, we have $f_y'(x, 0) = -\cos 1/x$. It follows that at the origin f_x' is discontinuous in y, and f_y' is discontinuous in x. However, the given function is totally differentiable at $(0, 0)$; for, we have

$$\underset{\substack{\Delta x \doteq 0 \\ \Delta y \doteq 0}}{\mathrm{L}} \left[\frac{f(\Delta x, \Delta y) - f(0, 0) - \Delta x f_x'(0, 0) - \Delta y f_y'(0, 0)}{\Delta(x, y)} \right]$$

$$= \underset{\substack{\Delta x \doteq 0 \\ \Delta y \doteq 0}}{\mathrm{L}} \left[\sqrt{(\Delta x)^2 + (\Delta y)^2} \sin \frac{1}{\Delta x + \Delta y} \right] = 0.$$

A PROPERTY OF CYCLOTOMIC INTEGERS AND ITS RELATION TO FERMAT'S LAST THEOREM.*

By H. S. Vandiver.

If p is an odd prime, and

$$x^p + y^p + z^p = 0 \tag{1}$$

is satisfied in rational integers prime to each other then Furtwängler† has shown that

$$\frac{r^{p-1} - 1}{p} = q(r) \equiv 0 \pmod{p},$$

for each factor r of x in case $x \not\equiv 0 \pmod{p}$ and for each factor r of $x^2 - y^2$ in case $x^2 - y^2$ is prime to p. Kummer‡ showed that if (1) is satisfied in integers prime to each other and to p, then

$$B_n \left[\frac{d^{p-2n} \log (x + e^v y)}{dv^{p-2n}}\right]_{v=0} \equiv 0 \pmod{p}, \tag{1a}$$

where $B_1 = 1/6$, $B_2 = 1/30$, $\cdots$ are the numbers of Bernoulli, and $n = 1$, $2, 3, \cdots (p-3)/2$. Mirimanoff§ proved that these criteria may be replaced by

$$B_n f_{p-2n}\left(-\frac{x}{y}\right) \equiv 0 \pmod{p}, \qquad \left(n = 1, 2, 3, \cdots \frac{p-3}{2}\right),$$

where

$$f_i(t) = t + 2^{i-1}t^2 + \cdots + (p-1)^{i-1}t^{p-1}.$$

In the present paper I shall derive the above results by methods a bit different from those employed by the writers mentioned, as well as some other criteria in reference to (1). It is shown that the criteria of Kummer and Furtwängler may all be derived from one relation.

1. Kummer|| proved that if $\mathfrak{P}$ is a prime ideal of the first degree in the algebraic field defined by $\alpha = e^{2i\pi/p}$ $(i = \sqrt{-1})$, then $\Pi_q \Pi_s \mathfrak{P}_{[1:s]}$ is a principal ideal in $\Omega(\alpha)$ where s ranges over the integers which satisfy

$$\frac{qp}{k+1} < s < \frac{qp}{k},$$

* Presented to the American Mathematical Society, Dec. 1914.

† Sitsungsberichte K. Akademie der Wissenschaften, Wien, vol. 121 (1912), p. 589.

‡ Abhandlungen der K. Akademie der Wissenschaften zu Berlin, 1857.

§ Journal für die Mathematik, vol. 128, pp. 45–68.

|| Journal für die Mathematik, vol. 35, p. 364.

q ranges over the set $1, 2, \cdots, k$, and k is any integer subject to the conditions $0 < k < p - 1$. Further $\mathfrak{P}_a$ is the ideal obtained from $\mathfrak{P}$ by the substitution (α/α^a) and $[1:s]$ is the least positive solution of $sX \equiv 1 \pmod{p}$. In Hilbert's* notation this may be expressed

(2)
$$\prod_q \prod_s \mathfrak{P}_{[1:s]} \sim 1.$$

Let $k = 1$, the relation becomes

(2a)
$$\prod_{s=(p+1)/2}^{p-1} \mathfrak{P}_{[1:s]} \sim 1.$$

Let $k = 2$ in (2) and multiply the resulting relation by (2a). The product may be written

(2b)
$$\prod_{\nu=1}^{2} \prod_{r=[\nu p/3]+1}^{p-1} \mathfrak{P}_{[1:r]} \sim 1.$$

Using the substitution (α/α^{p-1}), we get

$$\prod_{\nu=1}^{2} \prod_{r=1}^{[\nu p/3]} \mathfrak{P}_{[1:r]} \sim 1.$$

In like manner, comparison of (2b) with (2) for $k = 3$ gives

$$\prod_{\nu=1}^{3} \prod_{r=1}^{[\nu p/4]} \mathfrak{P}_{[1:r]} \sim 1,$$

and we find in general

(3)
$$\prod_{\nu=1}^{k-1} \prod_{r=1}^{[\nu p/k]} \mathfrak{P}_{[1:r]} \sim 1 \qquad (k = 2, 3, \cdots, p-1).$$

2. If $z \not\equiv 0 \pmod{p}$, we obtain from (1)

(4)
$$x + y = w^p, \qquad (w, \text{ an integer})$$

(5)
$$x + \alpha y = \mathfrak{q}^p,$$

where $\mathfrak{q}$ is an ideal in the field $\Omega(\alpha)$. If $\mathfrak{q}_s$ denotes the ideal obtained from $\mathfrak{q}$ by the substitution (α/α^s), we have from (5)

$$\Pi'((x + \alpha^{[1:r]}y)) = \Pi'\mathfrak{q}^p_{[1:r]},$$

if

$$\Pi' = \prod_{\nu=1}^{k-1} \prod_{r=1}^{[\nu p/k]}.$$

Since $\mathfrak{q}$ contains as factors only ideals of the first degree, the relation (3) gives

(6)
$$\Pi'(x + \alpha^{[1:r]}y) = \epsilon(\alpha)\theta^p,$$

where $\epsilon(\alpha)$ is a unit and θ is an integer in $\Omega(\alpha)$. Using the substitution (α/α^{-1}), we get

(6a)
$$\Pi'(x + \alpha^{-[1:r]}y) = \epsilon(\alpha^{-1})\theta^p_{-1}.$$

Now (1), (4), (6) and (6a) show that the ideal $(\theta\theta_{-1}) = ((-z/w)^{k-1})$, and

* Bericht der Deutschen Mathematiker Vereinigung, 1894, p. 223.

therefore $\theta\theta_{-1} = E(-z/w)^{k-1}$, where E is a unit in Ω. The product of (6) and (6a) then gives

(7) $$\epsilon(\alpha)\epsilon(\alpha^{-1})E^p = 1.$$

But we also have

(7a) $$\epsilon(\alpha) = \alpha^q\epsilon'(\alpha + \alpha^{-1}),^*$$

$\epsilon'(\alpha + \alpha^{-1})$ denoting a unit in the field defined by $\alpha + \alpha^{-1}$. From (7) and (7a) we obtain $(\epsilon'(\alpha + \alpha^{-1}))^2 = E^{-p}$, and if s and r are integers such that $2s = 1 + rp$, then

$$\epsilon'^{2s} = \epsilon'\epsilon'^{rp} = E^{-sp}$$

and $\epsilon' = E_1{}^p$, where E_1 is a unit in Ω. Hence $\epsilon = E_1{}^p\alpha^g$, and we obtain

(8) $$\alpha^{-g}\Pi'(x + \alpha^{[1:r]}y) = \omega^p.$$

Now by following a method employed by Kummer in the article cited, any integral relation between the pth roots of unity, say

$$f(\alpha) = 0,$$

may be replaced by

$$f(u) = F(u)\frac{u^p - 1}{u - 1},$$

where u is an arbitrary magnitude and $F(u)$ is an integral function of u. Applying this principle to (8), we obtain

(9) $$u^{-g}\Pi'(x + u^{[1:r]}y) = (\omega(u))^p + V(u)\frac{u^p - 1}{u - 1},$$

where $V(u)$ is an integral function of u. Setting $u = e^v$, we obtain from (9)

$$-vg + \sum_{v,r}\log(x + e^{v[1:r]}y) = \log(\omega^p + VX),$$

where

$$X = \frac{e^{vp} - 1}{e^v - 1}.$$

Put $[1:r] = n_r$. Differentiation gives

(10) $$-g + \sum_{v,r}\frac{d\log(x + e^{vn_r}y)}{dv} = \frac{d\log(\omega^p + VX)}{dv}.$$

Let $v = 0$, we get

(10a) $$-g + \frac{y}{x + y}\sum_{v,r}n_r = \left[\frac{d\log(\omega^p + VX)}{dv}\right]_{v=0}$$
$$= \left[\frac{1}{\omega^p + VX}\left(\frac{p\omega^{p-1}d\omega}{dv} + \frac{dVX}{dv}\right)\right]_{v=0}.$$

Now $[\omega^p + VX]_{v=0}$ is prime to p, since ω is prime to p. Further

$$\left[\frac{dVX}{dv}\right]_{v=0} \equiv 0 \pmod p,$$

* Hilbert, l. c., p. 336.

since $[X]_{v=0} = p$, and

$$\left[\frac{dX}{dv}\right]_{v=0} = 1 + 2 + 3 + \cdots + p - 1.$$

Hence

$$-g + \frac{y}{x+y}\sum_{\nu, r} n_r \equiv 0 \pmod{p}. \tag{11}$$

We note further that

$$\left[\frac{d^hX}{dv^h}\right]_{v=0} = 1 + 2^h + \cdots + (p-1)^h \equiv 0 \pmod{p},$$

for $0 < h < p - 1$. We therefore find

$$\left[\frac{d^h \log(\omega^p + VX)}{dv^h}\right]_{v=0} \equiv 0 \pmod{p},$$

since $[\omega^p + VX]_{v=0}$ is prime to p. Differentiation of (10) then gives

$$\sum_{\nu, r} n_r{}^h \left[\frac{d^h \log(x + e^v y)}{dv^h}\right]_{v=0} \equiv 0 \pmod{p}, \tag{12}$$

for $h = 2, 3, \cdots, p - 2$. We have, evidently,

$$\sum_{\nu, r} n_r{}^h \equiv \sum_{\nu=1}^{k-1}\sum_{j=1}^{[\nu p/k]} \frac{1}{j^h} \pmod{p}. \tag{13}$$

Now if $-y/x = t$ ($x \not\equiv 0 \pmod{p}$), then

$$\frac{d \log(x + e^v y)}{dv} = -\frac{te^v}{1 - te^v}.$$

We note that

$$\frac{d}{dv}\left(\frac{s}{1-s}(s^p - 1)\right) = (s^p - 1)\frac{d}{dv}\left(\frac{s}{1-s}\right) + \frac{s}{1-s}\frac{d}{dv}(s^p - 1),$$

where $s = e^v t$. Hence*

$$\left[\frac{d}{dv}\left(\frac{s(s^p-1)}{1-s}\right)\right]_{v=0} \equiv \left[(s^p - 1)\frac{d}{dv}\left(\frac{s}{1-s}\right)\right]_{v=0} \pmod{p},$$

since $1 - t$ is prime to p.

Proceeding in this way we find

$$\left[\frac{d^{a-1}}{dv^{a-1}}\left(\frac{s(s^p-1)}{1-s}\right)\right]_{v=0} \equiv (t^p - 1)\left[\frac{d^{a-1}}{dv^{a-1}}\left(\frac{s}{1-s}\right)\right]_{v=0},$$

modulo p. Or

$$f_a(t) = t + 2^{a-1}t^2 + \cdots + (p-1)^{a-1}t^{p-1}$$
$$\equiv -(t^p - 1)\left[\frac{d^{a-1}}{dv^{a-1}}\left(\frac{s}{1-s}\right)\right]_{v=0} \pmod{p}.$$

* Frobenius, Berlin Sitsungsberichte, July, 1910, p. 843.

Applying this to (12) and (13), we obtain

$$\sum_{\nu=1}^{k-1} \sum_{j=1}^{[\nu p/k]} \frac{1}{j^a} f_a(t) \equiv 0 \pmod{p}, \tag{14}$$

for $a = 2, 3, \cdots, p-2$.

3. To deduce the criteria of Kummer it is necessary to transform (13). Let m be an integer such that $1 < m < p$ and let $[k]$ represent the least positive residue of k_1 modulo p, where k is an integer. We have

$$mn = [mn] + mn - [mn]$$

and

$$m^s n^s \equiv [mn]^s + s[mn]^{s-1}(mn - [mn]) \pmod{p^2},$$

since $mn - [mn]$ is divisible by p. We have also

$$s[mn]^{s-1}(mn - [mn]) \equiv s(mn)^{s-1}(mn - [mn]) \pmod{p^2},$$

and therefore, if we let n range over the integers $1, 2, \cdots, p-1$, we get

$$\begin{aligned} m^s \Sigma n^s - \sum_n [mn]^s &= (m^s - 1)\Sigma n^s \\ &\equiv s m^{s-1} \sum_n n^{s-1}(mn - [mn]) \pmod{p^2}. \end{aligned} \tag{15}$$

Now, if $1 < s < p-1$,

$$\frac{m^s - 1}{s m^{s-1}} \frac{\Sigma n^s}{p} \equiv \sum_n \frac{n^{s-1}(mn - [mn])}{p} \pmod{p}.$$

If $b_{2a-1} = 0$, $b_{2a} = (-1)^{a-1} B_a$, it is known that

$$\Sigma n^s \equiv b_s p \pmod{p^2}.$$

We then have

$$\begin{aligned} \frac{b_s(m^s - 1)}{s m^{s-1}} &\equiv \sum_n \frac{n^{s-1}(mn - [mn])}{p} \pmod{p} \\ &\equiv \sum_n \left[\frac{mn}{p}\right] n^{s-1} \pmod{p} \\ &\equiv -\sum_n \left[\frac{mn}{p}\right] (p-n)^{s-1} \pmod{p} \end{aligned} \tag{16}$$

$$\equiv -\sum_{\nu=1}^{m-1} \sum_{j=1}^{[\nu p/m]} j^{s-1} \pmod{p}. \tag{16a}$$

Now from (15) for $s = p-1$ we obtain

$$-m \frac{m^{p-1} - 1}{p} \sum_n n^{p-1} \equiv \sum_n \frac{n^{p-2}(mn - [mn])}{p} \pmod{p},$$

and, noting that $\Sigma n^{p-1} \equiv -1 \pmod{p}$, we find as in the case $s < p-1$,

$$-mq(m) \equiv \sum_{\nu=1}^{m-1} \sum_{j=1}^{[\nu p/m]} \frac{1}{j} \pmod{p}.* \tag{17}$$

* For other derivations of (16a) and (17) see the author's article in these Annals, second series, vol. 18 (1917), p. 114.

Applying (16) to (14), we have

$$\frac{b_a(1-k^a)}{ak^{a-1}} f_{p-a}(t) \equiv 0 \pmod{p},$$

$k = 2, 3, \cdots, p-1$ and $a = 2, 3, \cdots, p-2$. Let k be a primitive root of p; then $1 - k^a \not\equiv 0 \pmod{p}$, and

$$b_a f_{p-a}(t) \equiv 0 \pmod{p},$$

which for a even gives the criteria of Kummer. Applying (13) and (17) to (11), we have

$$g \equiv -\frac{kyq(k)}{x+y} \pmod{p},$$

and we may state the result:

If (1) *is satisfied in integers prime to each other and z is prime to p, then it is necessary and sufficient that**

$$\prod_{\nu=1}^{k-1} \prod_{r=1}^{[\nu p/k]} (x + \alpha^{-1/r} y) = \alpha^{-[-kyq(k)]/(x+y)} \omega^p, \tag{20}$$

$$x + y = v^p \qquad (v \text{ an integer}), \tag{21}$$

where k is any integer $1 < k < p$, and ω is an integer in $\Omega(\alpha)$.

These conditions have already been proved necessary. To prove them sufficient make the substitution (α/α^{-1}) in (20) and multiply the resulting relation by (20) and the $(k-1)$th power of (21). We obtain

$$(x^p + y^p)^{k-1} = z_1^{\,p},$$

from which (1) follows.

4. To derive Furtwängler's theorem by means of (20) set

$$\zeta^{r^f-1/p} \equiv \left\{ \frac{\zeta}{\mathfrak{P}} \right\} \pmod{p},$$

where ζ is an integer in $\Omega(\alpha)$, $\mathfrak{P}$ is an ideal prime in $\Omega(\alpha)$ which is prime to ζ and p, r^f is the norm of $\mathfrak{P}$, and $\{\zeta/\mathfrak{P}\}$ is a certain pth root of unity. Then ζ is a perfect pth power, modulo $\mathfrak{P}$, if and only if

$$\left\{ \frac{\zeta}{\mathfrak{P}} \right\} = 1.$$

Let c be a rational prime factor of y in (20), if $y \not\equiv 0 \pmod{p}$. Then from (20),

$$\left\{ \frac{x}{c} \right\}^{\mu(k-1)} = \left\{ \frac{\alpha}{c} \right\}^{[-kyq(k)]/(x+y)}, \qquad \left(\mu = \frac{p-1}{2}\right), \tag{22}$$

where

$$\left\{ \frac{x}{c} \right\} = \left\{ \frac{x}{\mathfrak{P}_1} \right\} \left\{ \frac{x}{\mathfrak{P}_2} \right\} \cdots \left\{ \frac{x}{\mathfrak{P}_e} \right\},$$

$$c = \mathfrak{P}_1 \mathfrak{P}_2 \cdots \mathfrak{P}_e, \tag{23}$$

* Throughout this paper, whenever a fraction appears as an exponent of α, that exponent is the integer d, where d satisfies $df_1 \equiv f_2 \pmod{p}$, f_2/f_1 denoting the fraction in question.

the $\mathfrak{P}$'s being prime ideals in $\Omega(\alpha)$. Evidently

$$\{x/c\} = 1, \tag{24}$$

since if $\{x/c\} = \alpha^a$ $(a \neq 0)$, then by the substitution (α/α^m) (m prime to p) we obtain $\alpha^a = \alpha^{ma}$, whence $a = 0$. Let $k = p - 1$ in (22). We have $q(p-1) \not\equiv 0 \pmod{p}$, and since $y \not\equiv 0 \pmod{p}$, we have using (24)

$$\left\{\frac{\alpha}{c}\right\}^y = 1 = \left\{\frac{\alpha}{c}\right\}.$$

Now in (23) the $\mathfrak{P}$'s are distinct prime ideals each of degree f, where $ef = p - 1$.* Hence

$$\left\{\frac{\alpha}{c}\right\} = \left\{\frac{\alpha}{\mathfrak{P}_1}\right\} \cdots \left\{\frac{\alpha}{\mathfrak{P}_e}\right\}$$
$$= \alpha^{e\frac{c^f-1}{p}} = 1.$$

Hence

$$\frac{c^f - 1}{p} \equiv 0 \pmod{p}, \qquad q(c) \equiv 0 \pmod{p}.$$

We have

$$x + \alpha^s y = x \pm y + (\alpha^s \mp 1)y.$$

Substituting this form in (20) we get, provided that d is a factor of $x \pm y$,

$$\Pi' \left\{\frac{(\alpha^{[1:r]} \mp 1)y}{d}\right\} = \left\{\frac{\alpha}{d}\right\}^y,$$

or, by use of (17),

$$\left\{\frac{y}{d}\right\}^{(k-1)\mu} \left\{\frac{\alpha}{d}\right\}^{-\frac{kq(k)}{2}} \Pi' \left\{\frac{\alpha^{\frac{[1:r]}{2}} \mp \alpha^{-\frac{[1:r]}{2}}}{d}\right\} = \left\{\frac{\alpha}{d}\right\}^y. \tag{25}$$

The substitution (α/α^{-1}) shows that

$$\left\{\frac{\alpha^a \mp \alpha^{-a}}{d}\right\} = 1 \qquad (a \not\equiv 0 \pmod{p}).$$

Hence (25) reduces to

$$\left\{\frac{\alpha}{d}\right\}^{\frac{k(x-y)q(k)}{2(x+y)}} = 1.$$

If $k = p - 1$, this gives, provided that $x - y \not\equiv 0 \pmod{p}$,

$$\{\alpha/d\} = 1,$$

and, as before, we find $q(d) \equiv 0 \pmod{p}$. Whence the theorem:

If

$$\frac{x^p + y^p}{x + y} = v^p,$$

where x, y and v are integers prime to each other, then $q(v) \equiv 0 \pmod{p}$,

* Hilbert, l. c., p. 329.

where r is any factor of xy ($xy \not\equiv 0 \pmod p$) *and* $q(r) \equiv 0 \pmod p$ *for any factor r of* x^2-y^2, *provided that* x^2-y^2 *is prime to p.* Furtwängler's theorem follows from this.

5. Kummer in his first paper referred to above applied the relation (2) to (1) for the case where x, y and z are prime to p and found

$$\prod_s (x + \alpha^{[1:s]} y) = \alpha^a \gamma^p,$$

where s ranges as in (2) and γ is an integer in $\Omega(\alpha)$. The integer a was not determined by him. By methods already explained we find

$$a \equiv \frac{y}{x+y} \sum_s \frac{1}{s} \pmod p,$$

and*

$$\Sigma \frac{1}{s} \equiv -kq(k) + (k+1)q(k+1) \pmod p.$$

6. If in the relation (1) xyz is prime to p, then apply Wieferich's criteria $q(2) \equiv 0 \pmod p$ to (20) for $k = 2$. We obtain

$$\prod_{r=1}^{[p/2]} (x + \alpha^{[1:r]} y) = \omega^p.$$

Similar relations may be derived by using the criteria

$$q(3) \equiv q(5) \equiv q(11) \equiv q(17) \equiv 0 \pmod p.\dagger$$

7. In (1), assume that z is divisible by p. By Furtwängler's theorem we note that each factor of x or y which is of the form $1 \pmod p$ is necessarily of the form $1 \pmod{p^2}$. Hence

$$x^p + z^p \equiv x + z \pmod{p^3},$$
$$y^p + z^p \equiv y + z \pmod{p^3}$$

and

$$x^p + y^p + 2z^p \equiv x + y + 2z \pmod{p^3}.$$

Since

$$x + y \equiv 0 \pmod{p^3},$$

we have

$$x^p + y^p + 2z^p \equiv 0 \equiv 2z \pmod{p^3}$$

or $z \equiv 0 \pmod{p^3}$. In the relation (1), x, y and z cannot all be odd. If x or y is even, then Furtwängler's theorem gives $q(2) \equiv 0 \pmod p$; and if $q(2) \equiv 0 \pmod p$, then z must be divisible by $2p^3$. By using in addition the part of Furtwängler's theorem referring to factors of $x^2 - y^2$, it may be shown that $z \equiv 0 \pmod{3p^3}$, unless $q(3) \equiv 0 \pmod p$.

* Vandiver, l. c., p. 114.

† Mirimanoff, Comptes Rendus, vol. 150 (1910), p. 206; Vandiver, Journal für die Mathematik, vol. 144 (1914), p. 314; Frobenius, Sitzungsberichte der Preuss. Akademie der Wissenschaften, vol. 22 (1914), p. 65.

SURFACES OF ROTATION IN A SPACE OF FOUR DIMENSIONS.

By C. L. E. Moore.

1. The obvious generalization of a surface of rotation in ordinary space is a surface left invariant by a rotation in four dimensions, a rotation being defined as a linear transformation of positive determinant preserving distance and leaving one point fixed. In general the path curve of a point by a rotation in four dimensions is not a circle.* There are however two distinct types of rotations for which these path curves are circles. A rotation in general leaves two completely perpendicular planes invariant, not fixed point for point, but only as planes.† A special rotation leaves one of these planes fixed point for point. For this rotation the path curves are circles whose centers are in the second fixed plane and whose planes are parallel to the absolutely fixed plane. Surfaces generated by this kind of rotation have been studied somewhat.‡ If the rate of rotation in each of the invariant planes is the same the path curves are circles having their centers at the origin.§ In this note I shall confine my attention mostly to surfaces left invariant by this last kind of rotation. The path curves now however do not lie in parallel planes.

A general rotation in a space of four dimensions can be written

$$(1)\qquad \begin{aligned} X_1 &= x_1 \cos m_1t - x_2 \sin m_1t, & X_2 &= x_1 \sin m_1t + x_2 \cos m_1t, \\ X_3 &= x_3 \cos m_2t - x_4 \sin m_2t, & X_4 &= x_3 \sin m_2t + x_4 \cos m_2t, \end{aligned}$$

where the invariant planes are the X_1X_2- and the X_3X_4-planes, and m_1, m_2 are the rates of rotation in these planes. If the X_3X_4-plane is absolutely fixed, $m_2 = 0$. Surfaces generated by this rotation are the ones discussed by Wilson and Moore. The rotations we shall study now are given by $m_1 = \pm m_2$, but for the sake of simplicity we shall confine our attention to the case $m_1 = m_2 = 1$. The equations of the rotation then become

$$(2)\qquad \begin{aligned} X_1 &= x_1 \cos t - x_2 \sin t, & X_3 &= x_3 \cos t - x_4 \sin t, \\ X_2 &= x_1 \sin t + x_2 \cos t, & X_4 &= x_3 \sin t + x_4 \cos t. \end{aligned}$$

* See C. L. E. Moore, Motions in hyperspace, Annals of Math., second series, vol. 19 (1918), pp. 176–184.

† Cole, On rotations in four dimensions, American Journal of Mathematics, vol. 12 (1889), pp. 191–210.

‡ Wilson and Moore, Differential geometry of two-dimensional surfaces in hyperspace, Proceedings of the American Academy of Arts and Sciences, vol. 52 (1916), pp. 267–368.

§ Moore, Rotations in hyperspace, Proceedings of the American Academy of Arts and Sciences, vol. 53 (1918), pp. 649–694.

From (2) it is at once evident that

$$\sum_1^4 X_i^2 = \sum_1^4 x_i^2.$$

Consequently distance from the origin is left unchanged. Also if we allow t to vary, the curve which the point (X_1, X_2, X_3, X_4) describes will lie in the plane

$$(3)\qquad \begin{aligned} AX_1 + BX_2 + CX_3 + DX_4 &= 0, \\ BX_1 - AX_2 + DX_3 - CX_4 &= 0, \end{aligned}$$

where A, B, C, D are so chosen that

$$\begin{aligned} Ax_1 + Bx_2 + Cx_3 + Dx_4 &= 0, \\ Bx_1 - Ax_2 + Dx_3 - Cx_4 &= 0. \end{aligned}$$

Consequently if (x_1, x_2, x_3, x_4) lies in the plane (3), the curve (2) must lie in the same plane. Hence *any plane* (3) *is left invariant by the rotation* (2), *and consequently the path curve of a point lying in that plane must be a circle with center at the origin.*

The plane (3) cuts the planes

$$(4)\qquad \left|\begin{aligned} X_1 + iX_2 &= 0, \\ X_3 + iX_4 &= 0, \end{aligned}\right. \qquad \left|\begin{aligned} X_1 - iX_2 &= 0, \\ X_3 - iX_4 &= 0, \end{aligned}\right.$$

in lines for every value of A, B, C, D. The planes left invariant by the rotation form the linear congruence of planes cutting the planes (4). This congruence will contain the plane completely perpendicular to any plane (3) as can easily be verified. A linear congruence of planes, in space of four dimensions, which pass through a fixed point is simply isomorphic with a linear congruence of lines in ordinary space. We see then that one and only one plane of (3) contains any line which passes through the origin unless the given line lies in one of the planes (4).

2. **Rotation of a plane.** If a plane π passes through O, it will cut ∞^1 planes of (3) in lines. These planes then cut three fixed planes in lines and hence form one system of generators of a quadric hypercone which has O for simple vertex. The different positions which π takes will form the second system of generators of the cone. If the plane does not pass through O, let its equations be

$$(5)\qquad \sum_1^4 A_i X_i + A_5 = 0, \qquad \sum_1^4 B_i X_i + B_5 = 0.$$

If we apply the transformation (2) to this plane, we have on eliminating t

$$(6)\qquad \begin{vmatrix} A_5 & \lambda_1 \\ B_5 & \lambda_2 \end{vmatrix}^2 + \begin{vmatrix} \mu_1 & A_5 \\ \mu_2 & B_5 \end{vmatrix}^2 = \begin{vmatrix} \lambda_1 & \mu_1 \\ \lambda_2 & \mu_2 \end{vmatrix}^2,$$

where

$$\lambda_1 = \sum_1^4 A_i x_i, \qquad \lambda_2 = \sum_1^4 B_i x_i,$$
$$\mu_1 = A_2x_1 - A_1x_2 + A_4x_3 - A_3x_4,$$
$$\mu_2 = B_2x_1 - B_1x_2 + B_4x_3 - B_3x_4.$$

Hence *the hypersurface generated by revolving a plane not passing through the origin is of order four and has a double point at the origin.* If the plane passes through O, $A_5 = B_5 = 0$ and (6) reduces to a quadric cone. If λ_1, λ_2, μ_1, μ_2 have a line in common, the cone will have this common line for vertex. But we see that $\lambda_1 = 0$, $\mu_1 = 0$ forms a plane of (3), and likewise $\lambda_2 = 0$, $\mu_2 = 0$. Two planes (3) can only intersect in a line lying in one of the planes (4). Hence if a plane cutting one of the planes (4) in a line is rotated, it will generate a cone having this line for vertex. These cones are evidently imaginary.

The general hypersurface of order four above will cut each of the invariant planes in a circle, since two planes in space of four dimensions always have a point in common and the point will rotate in a circle. However, if the plane passes through O, the quadric cone generated will cut only a simple infinity of the invariant planes in circles but will cut each of these in an infinite number of concentric circles.

Two near by positions of the plane (5) will have a point in common. The locus of this point will be a circle to which all the planes are tangent. For the plane near to (5) is

$$(7)\quad \begin{cases} A_1(x_1 - x_2dt) + A_2(x_1dt + x_2) + A_3(x_3 - x_4dt) + A_4(x_3dt + x_4) = 0, \\ B_1(x_1 - x_2dt) + B_2(x_1dt + x_2) + B_3(x_3 - x_4dt) + B_4(x_3dt + x_4) = 0. \end{cases}$$

By means of (5) equations (7) are reducible to

$$(8)\quad \begin{cases} A_2x_1 - A_1x_2 + A_4x_3 - A_3x_4 = 0, \\ B_2x_1 - B_1x_2 + B_4x_3 - B_3x_4 = 0. \end{cases}$$

In equations (2) we consider x_i as fixed; this will be the point corresponding to $t = 0$. The point corresponding to the value $t = dt$ will be

$$(x_1 - x_2dt,\ x_1dt + x_2,\ x_3 - x_4dt,\ x_3dt + x_4),$$

and equation (7) is the condition that this point lie in (5). Consequently the plane (5) will be tangent to the locus of intersection of (5) and (8). If (5) and (8) have more than one point in common, they will evidently have a straight line in common. Hence *if a plane is rotated it will always be tangent to one of the invariant circles (path curves), and if it is tangent to more than one of them it will be tangent to an infinite number and the points of tangency will all lie on a straight line.*

If (5) and (8) have a line in common, the hypercone (6) will have this line for vertex.

3. **Rotation of a line.** If a line passing through O is rotated, it will generate the invariant plane containing this line, or if the line lies in an invariant plane and does not pass through O it will generate this plane when rotated. If the line does not pass through O nor lie in an invariant plane, two planes can be passed through it, one containing O and the other not. Then the surface formed by rotating the line will be the intersection of the hypersurfaces formed by rotating these two planes. Hence *the surface formed by rotating a line is of order eight and will have a multiple point at the origin.* The only developable surfaces (that is generated by the tangent lines to a curve) are planes.

If in (2) we make x_i functions of s (arc length), equations (2) become the parametric equations of the surface formed by rotating the curve

$$x_i = x_i(s) \qquad (i = 1, 2, 3, 4). \tag{9}$$

The vector equation of the surface is

$$\rho = \Sigma X_i k_i. \tag{10}$$

Then

$$\left\{\begin{aligned} m = \frac{\partial \rho}{\partial s} &= (x_1' \cos t - x_2' \sin t)k_1 + (x_1' \sin t + x_2' \cos t)k_2 \\ &\quad + (x_3' \cos t - x_4' \sin t)k_3 + (x_3' \sin t + x_4' \cos t)k_4, \\ n = \frac{\partial \rho}{\partial t} &= -(x_1 \sin t + x_2 \cos t)k_1 + (x_1 \cos t - x_2 \sin t)k_2 \\ &\quad - (x_3 \sin t + x_4 \cos t)k_3 + (x_3 \cos t - x_4 \sin t)k_4, \end{aligned}\right. \tag{11}$$

where primes indicate derivatives with respect to s.

Now

$$m \cdot n = x_1'x_2 - x_2'x_1 + x_3'x_4 - x_4'x_3. \tag{12}$$

Hence the condition that the curve (9) be orthogonal to the path curves cutting it is

$$x_1'x_2 - x_2'x_1 + x_3'x_4 - x_4'x_3 = 0. \tag{13}$$

If (9) is a straight line, x_i will be linear functions of s

$$x_i = a_i s + b_i,$$

and we obtain on substituting in (13)

$$a_1 b_2 - a_2 b_1 + a_3 b_4 - a_4 b_3 = 0, \tag{14}$$

as the condition that the rulings on a ruled surface of rotation cut the path curves orthogonally. Since (14) does not contain the parameter s, we see that if one path curve cuts a straight line at right angles, every path curve cutting it will cut it at right angles. Hence passing through each point of space are ∞^2 lines which are orthogonal to every path curve they cut, viz., the lines passing through the point and lying in the space of

three dimensions perpendicular to the path curve through the point. In each plane there is a pencil of such lines. Also if two intersecting lines are orthogonal to the path curve through the point of intersection, each line of the pencil formed by them will have the same property.

4. **Rotation of plane curves.** If (9) defines a curve lying in the plane

$$\begin{aligned} x_1 &= A_1x_2 + B_1x_4 + C_1, \\ x_3 &= A_2x_2 + B_2x_4 + C_2, \end{aligned} \tag{15}$$

then the condition that the curve be orthogonal to the path curves is

$$(B_1 - A_2)(x_2x_4' - x_2'x_4) - C_1x_2' - C_2x_4' = 0. \tag{16}$$

The solution of this equation is

$$(B_1 - A_2)x_4 - C_1 = k[(B_1 - A_2)x_2 - C_2].$$

This equation together with (15) determines a pencil of lines in (15) which are the only curves lying in a general plane which cut the path curves orthogonally. However if (16) is identically satisfied $B_1 = A_2$, $C_1 = C_2 = 0$, and (15) becomes

$$\begin{aligned} x_1 &= A_1x_2 + B_1x_4, \\ x_3 &= B_1x_2 + B_2x_4. \end{aligned} \tag{17}$$

Then every path curve cutting (17) cuts it orthogonally. It is easily seen that the plane passing through O and completely perpendicular to (3) is

$$\begin{aligned} (AC + BD)x_1 + (BC - AD)x_2 - (A^2 + B^2)x_3 &= 0, \\ (AD - BC)x_1 + (AC + BD)x_2 - (A^2 + B^2)x_4 &= 0. \end{aligned} \tag{18}$$

This plane is also seen to belong to the system of invariant planes. If the plane (17) cuts (3) in a line, it is easily shown that it will also cut (18) in a line. A general plane cutting (3) and (18) in lines must have two essential constants. Equation (17) has two essential constants and consequently is the general form of the equation of a plane which if it cuts one invariant plane in a line through the origin will also cut the invariant plane completely perpendicular to it in a line. That is the invariant planes cutting (17) can be grouped into completely perpendicular pairs. Hence *if a curve lies in a plane which cuts two completely perpendicular planes, it will be orthogonal to every path curve which cuts it.*

5. **Some particular surfaces.** The element of arc of the surface (10) is

$$d\sigma^2 = m \cdot m ds^2 + 2m \cdot n ds dt + n \cdot n dt^2.$$

If the generating curve (9) is orthogonal to the path curves,

$$m \cdot n = 0, \qquad m \cdot m = \Sigma x_i'^2 = 1, \qquad n \cdot n = \Sigma x_i^2$$

and the element of arc then becomes

$$d\sigma^2 = ds^2 + \Sigma x_i^2 dt^2.$$

The expression Σx_i^2 is a function of s only. Hence *any of these surfaces can be mapped without stretching on a surface of revolution in ordinary space.* If (10) is a developable surface, we know from the theory of surfaces of revolution in ordinary space that

$$\Sigma x_i^2 = (as + b)^2. \tag{19}$$

There are surfaces besides cylinders and cones which satisfy this relation in space of four dimensions. In fact we do not have cones or cylinders of revolution for this kind of rotation. *The only ruled developable surfaces of revolution are planes.* For the only ruled developable surfaces are those formed by the tangents to a twisted curve.* If the surface is one of rotation, the edge of regression must be left invariant; but the only invariant curves are circles, consequently the only surfaces of revolution of this type are planes.

An interesting class of developables are those generated by curves lying in planes cutting two completely perpendicular planes in lines. For simplicity let us take a curve lying in the x_1x_3-plane. This is not a specialization since any such plane can be brought into this position by a simple change of coördinates. Using polar coördinates in this plane, we show that the condition for a developable surface becomes

$$r = as + b,$$

or

$$dr = a\sqrt{dr^2 + r^2d\theta^2}.$$

The solution of this equation is

$$r = e^{\frac{a}{\sqrt{1-a^2}}(\theta-\theta_0)}.$$

Hence *logarithmic spirals lying in planes cutting two completely perpendicular planes in lines by this rotation generate developable surfaces.*

Formula (19) shows that any curve lying on a hypersphere with center at O will generate a developable surface. In particular a circle lying in the x_1x_3-plane and having center at O will generate a developable of revolution.

If a surface of revolution in a space of four dimensions can be mapped on a sphere in ordinary space, then

$$d\sigma^2 = a^2(d\theta^2 + \sin^2\theta d\varphi^2),$$

and if we make the transformation $\theta = s/a$, $\varphi = t$,

$$d\sigma^2 = ds^2 + a^2 \sin^2 \frac{s}{a} dt^2.$$

* Wilson and Moore, loc. cit., p. 342.

This again represents a large class of surfaces but we will restrict the curve as above to lie in the x_1x_3-plane. Using polar coördinates again, we have

$$r = a \sin \frac{s}{a}, \qquad \text{or} \qquad s = a \sin^{-1} \frac{r}{a},$$

$$dr^2 + r^2 d\theta^2 = \frac{a^2 dr^2}{a^2 - r^2}.$$

The solution of this equation is

$$r = a \sin (\theta + \theta_0).$$

Hence, *circles passing through the origin and lying in a plane cutting a pair of completely perpendicular planes in lines will generate a surface of revolution which can be mapped on a sphere in a space of three dimensions.*

The surface formed by rotating a curve C lying in the x_1x_3-plane has two systems of plane sections. The planes of these two sets of curves form the two generations of a quadric hypercone with vertex at O. A plane of one system contains a perpendicular to each plane of the other system. If the plane of C passes through O but does not cut two completely perpendicular invariant planes, the planes of the two sets of plane sections still form the two generations of a quadric cone, but there is no perpendicularity relation between the planes. If the plane of C does not pass through O, one system of planes (those containing C) generate a quartic cone and the planes containing the path curves generate a cone with O as vertex, formed by projecting from O the ruled surface formed by the lines cutting C and a line in each of the planes (4). [These lines are obtained as the intersection of a space of three dimensions containing C and cutting the planes (4).] This ruled surface will, in general, be of order $2m$ where m is the order of C. Hence *the surface formed by rotating a plane curve of order m is, in general, of order $8m$.*

6. **Minimum surfaces.** The general condition that a surface be minimum is the vanishing of the vector mean curvature.* To express the vector mean curvature we need the three covariant derivatives and for this purpose I shall make use of the formulas developed by Wilson and Moore,†

$$(20) \qquad y_{11} = \frac{1}{a}\begin{vmatrix} m & n & p \\ m\cdot m & m\cdot n & m\cdot p \\ m\cdot n & n\cdot n & n\cdot p \end{vmatrix}, \qquad y_{12} = \frac{1}{a}\begin{vmatrix} m & n & q \\ m\cdot m & m\cdot n & m\cdot q \\ m\cdot n & n\cdot n & n\cdot q \end{vmatrix},$$

$$y_{22} = \frac{1}{a}\begin{vmatrix} m & n & r \\ m\cdot m & m\cdot n & m\cdot r \\ m\cdot n & n\cdot n & n\cdot r \end{vmatrix},$$

* Wilson and Moore, loc. cit., p. 325.

† Loc. cit., pp. 337–8.

where $a = (m \cdot m)(n \cdot n) - (m \cdot n)^2$.

$$(21)\quad \begin{cases} m = \dfrac{\partial \rho}{\partial s} = (x_1' \cos t - x_2' \sin t)k_1 + (x_1' \sin t + x_2' \cos t)k_2 \\ \qquad + (x_3' \cos t - x_4' \sin t)k_3 + (x_3' \sin t + x_4' \cos t)k_4, \\ n = \dfrac{\partial \rho}{\partial t} = -(x_1 \sin t + x_2 \cos t)k_1 + (x_1 \cos t - x_2 \sin t)k_2 \\ \qquad - (x_3 \sin t + x_4 \cos t)k_3 + (x_3 \cos t - x_4 \sin t)k_4, \\ p = \dfrac{\partial^2 \rho}{\partial s^2} = \dfrac{\partial m}{\partial s} = (x_1'' \cos t - x_2'' \sin t)k_1 + (x_1'' \sin t + x_2'' \cos t)k_2 \\ \qquad + (x_3'' \cos t - x_4'' \sin t)k_3 + (x_3'' \sin t + x_4'' \cos t)k_4, \\ q = \dfrac{\partial^2 \rho}{\partial s \partial t} = \dfrac{\partial m}{\partial t} = -(x_1' \sin t + x_2' \cos t)k_1 + (x_1' \cos t - x_2' \sin t)k_2 \\ \qquad - (x_3' \sin t + x_4' \cos t)k_3 + (x_3' \cos t - x_4' \sin t)k_4, \\ r = \dfrac{\partial^2 \rho}{\partial t^2} = \dfrac{\partial n}{\partial t} = -(x_1 \cos t - x_2 \sin t)k_1 - (x_1 \sin t + x_2 \cos t)k_2 \\ \qquad - (x_3 \cos t - x_4 \sin t)k_3 - (x_3 \sin t + x_4 \cos t)k_4. \end{cases}$$

Since s is the length of arc of the revolved curve

$$m \cdot m = \Sigma x_i'^2 = 1,$$

and by differentiating

$$\Sigma x_i' x_i'' = 0.$$

If we take the path curves perpendicular to the generating curve, that is take an orthogonal trajectory of the path curves for generating curve, we have also

$$m \cdot n = x_1' x_2 - x_2' x_1 + x_3' x_4 - x_4' x_3 = 0.$$

Differentiating this last relation, we also have

$$x_1'' x_2 - x_2'' x_1 + x_3'' x_4 - x_4'' x_3 = 0.$$

Using these relations, we find

$$(22)\quad \begin{cases} n \cdot n = \Sigma x_i^2, \quad m \cdot p = \Sigma x_i' x_i'' = 0, \quad m \cdot q = 0, \\ m \cdot r = -\Sigma x_i x_i', \quad n \cdot p = x_1 x_2'' - x_2 x_1'' + x_3 x_4'' - x_4 x_3'' = 0, \\ n \cdot q = \Sigma x_i x_i', \quad n \cdot r = 0, \quad a = n \cdot n = \Sigma x_i^2. \end{cases}$$

Substituting these values in (20), we have

$$(23)\qquad y_{11} = p, \qquad y_{12} = q - \frac{n \cdot q}{n \cdot n} n, \qquad y_{22} = r - (m \cdot r)m,$$

From (22) it is at once seen that y_{11}, y_{12}, y_{22} are normal to the surface which indeed is always true of the absolute derivatives. If the curvature vector of a curve, traced on a surface, lies in the normal plane to the surface the curve is a geodesic. Then from the first equation of (23) we see that

*the orthogonal trajections of the path curves of a surface of revolution in space of four dimensions are geodesics.** The last equation of (23) shows that $m \cdot r = 0$ is a sufficient condition that the path curves be geodesics. The integral of this is

$$\Sigma x_i^2 = a^2.$$

The general condition that the path curves be geodesics is that r lie in the normal plane, that is that r be perpendicular to m and n. From (22) $n \cdot r = 0$, consequently $m \cdot r = 0$ is both necessary and sufficient. Hence *the only surfaces of revolution on which the path curves are geodesics are those formed by revolving a spherical curve.* From (19) we see that all such surfaces are developable. If $m \cdot r = 0$, then $n \cdot q = 0$, and (23) shows that p, q, r are three normals to the surface and consequently satisfy a linear relation.

The vector mean curvature is given by the formula (Wilson and Moore, p. 322)

$$2h = \Sigma a^{(rs)} y_{rs} = p + \frac{r}{n \cdot n} - \frac{m \cdot r}{n \cdot n} m.$$

The vanishing of this vector is the condition for a minimum surface. Then the coördinates of the curve will have to satisfy the differential equations

$$(24) \qquad (\sum_i x_i^2) x_j'' + (\sum_i x_i x_i') x_j' - x_j = 0, \qquad j = 1, 2, 3, 4.$$

Multiply the first of these equations by x_2 and the secondly $-x_1$ and add, then the first by x_3 and the third by $-x_1$ and add, and finally the first by x_4 and the fourth by $-x_1$ and add. We thus obtain

$$(25) \qquad \begin{cases} (\Sigma x_i^2)(x_1'' x_2 - x_2'' x_1) + \Sigma x_i x_i' (x_1' x_2 - x_2' x_1) = 0, \\ (\Sigma x_i^2)(x_1'' x_3 - x_3'' x_1) + \Sigma x_i x_i' (x_1' x_3 - x_3' x_1) = 0, \\ (\Sigma x_i^2)(x_1'' x_4 - x_4'' x_1) + \Sigma x_i x_i' (x_1' x_4 - x_4' x_1) = 0. \end{cases}$$

Eliminating the Σ's, we have

$$\frac{x_1'' x_2 - x_2'' x_1}{x_1' x_2 - x_2' x_1} = \frac{x_1'' x_3 - x_3'' x_1}{x_1' x_3 - x_3' x_1} = \frac{x_1'' x_4 - x_4'' x_1}{x_1' x_4 - x_4' x_1}.$$

The first integral is

$$x_1' x_2 - x_2' x_1 = C_1(x_1' x_3 - x_3' x_1) = C_2(x_1' x_4 - x_4' x_1).$$

The complete integral then is

$$x_1 = A x_2 + B x_3,$$
$$x_2 = C x_2 + D x_4.$$

Hence if there exists a minimum surface it must be formed by rotating a plane curve but we saw that the only plane curves that are orthogonal to

* Cf. E. E. Levi, Sui gruppi di movimenti, Atti dei Lincei, Series 5, vol. XIV[1] (1905).

the path curves lie in planes cutting a pair of completely perpendicular invariant planes in lines. Then by a proper choice of axes we can take the curve in the x_1x_3-plane. Equations (24) then become

$$(x_1^2 + x_3^2)x_1'' + (x_1x_1' + x_3x_3')x_1' - x_1 = 0,$$
$$(x_1^2 + x_3^2)x_3'' + (x_1x_1' + x_3x_3')x_3' - x_3 = 0.$$

If we assume the relation

$$x_1'^2 + x_3'^2 = 1. \tag{26}$$

these equations are not independent, since then the sum of the first multiplied by x_1' and the second multiplied by x_3' is zero. On the assumption (26) the second equation reduces to

$$\frac{\dfrac{d^2x_3}{dx_1^2}}{1 + \left(\dfrac{dx_3}{dx_1}\right)^2} + \frac{x_1\dfrac{dx_3}{dx_1} - x_3}{x_1^2 + x_3^2} = 0.$$

The first integral is

$$\frac{dx_3}{dx_1} = \frac{C_1x_1 - x_3}{x_1 + C_1x_3}$$

and the complete integral is

$$C_1(x_3^2 - x_1^2) + 2x_1x_3 + C_2 = 0.$$

Hence *equilateral hyperbolas with center at O and lying in planes which cut a pair of completely perpendicular invariant planes in lines generate minimum surfaces by this rotation.* These surfaces and planes are the only minimum surfaces left invariant by the rotation (2).

If $n \cdot n$ = const., equation (25) no longer holds but in this case the surface is developable and there are no minimum developable surfaces. For if the surface is minimum, the center of the indicatrix is at the surface point,* and if it is developable, the distance from the center of the indicatrix to the surface point is $\sqrt{a^2 + b^2}$, where a and b are the axes of the indicatrix. Therefore if the surface is both minimum and developable, the indicatrix must reduce to a point. The only real surface for which this is true is a plane.

7. **Surfaces for which the indicatrix reduces to a linear segment.** Another interesting class of surfaces in hyperspace are those for which the indicatrix reduces to a linear segment because those surfaces possess many of the properties of surfaces in three dimensions. The area of the indicatrix is given by the formula

$$a_{12}y_{22}xy_{11} + a_{22}y_{11}xy_{12} + a_{11}y_{12}xy_{22},$$

* Wilson and Moore, loc. cit., p. 326.

where the cross indicates the Gibbs vector product and a_{rs} are the coefficients in the first fundamental form. Then

$$a_{11} = 1, \qquad a_{12} = 0, \qquad a_{22} = \Sigma x_i^2.$$

The vanishing of this invariant will then be the condition that the indicatrix reduce to a linear segment. Substituting the values of the quantities previously found this reduces to

$$[(\Sigma x_i^2)p - r + (m\cdot r)m] \times [q - (n\cdot q)n] = 0. \tag{27}$$

This product will vanish if either factor vanishes or if the two vectors are parallel. If the first factor vanishes, we have

$$(\Sigma x_i^2)x_j'' - (\Sigma x_i x_i')x_j' + x_j = 0, \qquad j = 1, 2, 3, 4. \tag{28}$$

Multiply by x_j and sum on j; we obtain

$$(\sum_i x_i^2)(\sum_j x_j x_j'' + 1) - (\sum_i x_i x_i')^2 = 0.$$

The complete integral is

$$\Sigma x_i^2 = (C_1 s + C_2)^2.$$

Hence the only surfaces for which the first factor of (27) vanishes are developables. The set of equations (28) when treated the same as we did (24) lead to the same set of equations showing that the surfaces must be generated by rotating plane curves whose plane cuts a pair of completely perpendicular invariant planes. If $m\cdot n$ = const., the argument does not apply. In this case the surface of rotation lies on a sphere and (27) becomes

$$(C^2 p - r) \times q = 0,$$

and if the first factor of this vanishes

$$C^2 x_j'' + x_j = 0, \qquad j = 1, 2, 3, 4.$$

The solution is

$$x_j = A_j \cos\frac{s}{c} + B_j \sin\frac{s}{c}.$$

This is a plane curve and therefore a circle and can be taken in the x_1x_3-plane.

If the second factor of (27) vanishes,

$$(\sum_i x_i^2)x_j' - (\Sigma x_i x_i')x_j = 0, \qquad j = 1, 2, 3, 4.$$

The solution is seen to be a straight line which by the rotation (2) forms a plane.

Finally if the factors are parallel vectors, we have

$$\frac{(\sum_i x_i^2)x_j'' - (\sum_i x_i x_i')x_j' + x_j}{(\Sigma x_i^2)x_j' - (\Sigma x_i x_i')x_j} = \text{const.}, \qquad j = 1, 2, 3, 4.$$

Multiplying the numerators and denominators respectively by x_1', x_2', x_3', x_4' and adding, we find the new numerator vanishes and consequently the denominators must vanish and we have

$$\Sigma x_i^2 - (\Sigma x_i x_i')^2 = 0.$$

The integral is

$$\Sigma x_i^2 = (s + c)^2.$$

Since Σx_i^2 is the distance from the origin to a point of the curve it is obvious that the only curves that will satisfy this relation are lines passing through the origin. *Besides planes the only surfaces of revolution for which the indicatrix reduces to a linear segment are developables formed by rotating plane curves.*

8. **Locus of the indicatrix.** The indicatrix is an ellipse lying in the normal plane and if the normal plane is revolved, in each position it will contain the indicatrix at the new point. Hence the locus of the indicatrix corresponding to points of a path curve on a surface of revolution is a surface of order 16 having one set of equal conics and one set of circles with center at O for plane sections. If the indicatrix reduces to a straight line the locus becomes a ruled surface of revolution and consequently of order 8. The end of the mean curvature vector will generate a surface of revolution on which the circles (path curves) will correspond to the circles on the original surface of revolution. The circles on this surface are the locus of the center of the indicatrix corresponding to circles of the original surface.

9. **The General rotation.** If instead of the special rotation (2) we use the general rotation (1) having the origin for fixed point, we obtain a surface of revolution having many of the characteristics of this special one. In this case however the path curves are not circles and indeed if m_1 and m_2 are incommensurable the path curves are not even closed curves. If we consider $m_1 : m_2$ and t as parameters we obtain the surface left invariant by the two parameter group having the same two fixed planes. This is a developable surface traced on the sphere.* The cartesian equations of this surface are

$$x_1^2 + x_2^2 = a^2, \qquad x_3^2 + x_4^2 = b^2.$$

Many of the properties of this surface are given in my article just referred to.

* Cf. C. L. E. Moore, "Rotations in hyperspace," Proceedings of the American Academy of Arts and Sciences, vol. 53 (1918), p. 676.

For this type of rotation there are ruled developable surfaces of revolution, viz., the surface formed by revolving the tangent lines to a path curve. The locus of a straight line or a plane are in this case transcendental.

For this general rotation surfaces traced on a sphere are not necessarily developable. The general developable satisfies the condition

$$m_1^2(x_1^2 + x_2^2) + m_2^2(x_3^2 + x_4^2) = (as + b)^2,$$

from which we see that surfaces traced on a sphere for the special rotation correspond here to the surfaces traced on the ellipsoid

$$m_1^2(x_1^2 + x_2^2) + m_2^2(x_3^2 + x_4^2) = b^2.$$

Two pairs of axes of this ellipsoid are equal.

The condition that the curve rotated be an orthogonal trajectory of the path curves is

$$m_1(x_1'x_2 - x_2'x_1) + m_2(x_3'x_4 - x_4'x_3) = 0.$$

Equations (23) remain the same from which we see that *the orthogonal trajectories of the path curves are geodesics.* Also the only surfaces for which the path curves are geodesics are the surfaces traced on the ellipsoid

$$m_1^2(x_1^2 + x_2^2) + m_2^2(x_3^2 + x_4^2) = b^2.$$

It is easily seen also that the indicatrix of surfaces traced on this ellipsoid always degenerates into a linear segment.

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THE CIRCLE NEAREST TO N GIVEN POINTS, AND THE POINT NEAREST TO N GIVEN CIRCLES.

By Julian Lowell Coolidge.

A. Suppose that we are given n points $(x_1, y_1)(x_2, y_2) \cdots (x_n, y_n)$. These points were expected to lie upon a circle, but, for some reason, they fail to do so. What circle shall we take as that which most nearly fulfills the condition of passing through them?

The problem, as so stated, is lacking in precision, owing to the obscurity of the words "most nearly." They may be given a precise meaning in various ways; if, however, we are at all under the sway of the dogma of least squares, it seems natural to state the problem exactly in the following terms:

To find a circle such that the sum of the squares of the distances from each of n given points to the nearest point of the circle shall be a minimum.[1]

Let (x, y) be the coördinates of the center of the required circle while r is its radius. Let us further assume, for simplicity of calculation, that the origin is at the center of gravity of the given points so that

$$\sum_i x_i = \sum_i y_i = 0 \qquad (i = 1, 2 \cdots n). \tag{1}$$

The condition imposed may be written

$$\Sigma[|\sqrt{(x - x_i)^2 + (y - y_i)^2}| - r]^2 = Minimum. \tag{2}$$

Let us write, for simplicity

$$|\sqrt{(x - x_i)^2 + (y - y_i)^2}| = d_i + r. \tag{3}$$

Equating to zero the three partial derivatives of (2) we have

$$\begin{aligned} \Sigma\, d_i &= 0, \\ \Sigma \frac{(x - x_i)d_i}{d_i + r} &= 0, \\ \Sigma \frac{(y - y_i)d_i}{d_i + r} &= 0. \end{aligned} \tag{4}$$

[1] Vahlen "Konstruktionen und Approximationen," Leipzig, 1911, p. 126, says: "Für die konstruktive Ausgleichung . . . eines Kreises aus mehr als drei Punkten scheinen dem Bertotschen entsprechende einfache Verfahren noch nicht gefunden zu sein." For $n = 4$ he finds the circle whose *greatest* distance from any given point shall be a minimum.

THEOREM. *The circle sought is characterized by the fact that the geometric sum of the vectors from the given points to the respectively nearest points of the circle is zero, while the radius is the average of the distances from the given points to the center.*

It is immediately apparent that it would be quite hopeless to undertake to solve equations (3) and (4) in their present form. We can, however, reach an approximate solution by assuming that the points are nearly concyclic so that d_i is small compared with r, *i.e.*, that $(d_i/r)^2$ is negligible. Then

$$\frac{d_i}{d_i + r} = \frac{d_i}{r}\left(1 + \frac{d_i}{r}\right)^{-1} = \frac{d_i}{r},$$

$$(5)\qquad \begin{aligned} \Sigma d_i = \Sigma(x - x_i)d_i = \Sigma(y - y_i)d_i &= 0, \\ \Sigma(x - x_i)^2 + \Sigma(y - y_i)^2 - nr^2 &= 0, \\ \Sigma(x - x_i)^3 + \Sigma(x - x_i)(y - y_i)^2 - r^2\Sigma(x - x_i) &= 0, \\ \Sigma(x - x_i)^2(y - y_i) + \Sigma(y - y_i)^3 - r^2\Sigma(y - y_i) &= 0. \end{aligned}$$

These equations are noteworthy for two reasons:

1. They are just the equations we should have reached, had we started from the assumption, which is nearly as natural as the one which we actually made, that the sum of the squares of the powers of the given points with regard to the circle sought should be a minimum.

2. Although apparently involving high powers of x and y they are quite easy to solve. Let us eliminate r^2 between the first and second, and between the first and third; we get

$$(7)\qquad \begin{aligned} &n\Sigma(x - x_i)^3 - \Sigma(x - x_i)^2\Sigma(x - x_i) + n\Sigma(x - x_i)(y - y_i)^2 \\ &\qquad - \Sigma(x - x_i)\Sigma(y - y_i)^2 = 0, \\ &n\Sigma(y - y_i)^3 - \Sigma(x - x_i)\Sigma(y - y_i)^2 + n\Sigma(x - x_i)^2(y - y_i) \\ &\qquad - \Sigma(x - x_i)^2\Sigma(y - y_i) = 0. \end{aligned}$$

These are reduced, with the aid of (1) to

$$(6)\qquad \begin{aligned} 2(\Sigma x_i^2)x + 2(\Sigma x_i y_i)y &= \Sigma x_i(x_i^2 + y_i^2), \\ 2(\Sigma x_i y_i)x + 2(\Sigma y_i^2)y &= \Sigma y_i(x_i^2 + y_i^2). \end{aligned}$$

We thus reach our final solution

$$(7)\qquad \begin{aligned} x &= \frac{\Sigma x_i(x_i^2 + y_i^2)\Sigma y_i^2 - \Sigma y_i(x_i^2 + y_i^2)\Sigma x_i y_i}{2[\Sigma x_i^2\Sigma y_i^2 - (\Sigma x_i y_i)^2]}, \\ y &= \frac{\Sigma y_i(x_i^2 + y_i^2)\Sigma x_i^2 - \Sigma x_i(x_i^2 + y_i^2)\Sigma x_i y_i}{2[\Sigma x_i^2\Sigma y_i^2 - (\Sigma x_i y_i)^2]}, \\ r &= \sqrt{\frac{\Sigma[(x - x_i)^2 + (y - y_i)^2}{n}}. \end{aligned}$$

B. Suppose that we are given n circles whose centers are the points $(x_1, y_2)(x_2, y_2) \cdots (x_n, y_n)$, while their respective radii are $r_1, r_2, \cdots, r_n$. These circles were supposed to pass through a common point, but failed to do so. What point shall we take to replace their supposed point of concurrence?

The first stages of this problem are surprisingly like those of *A*. If we seek a point such that the sum of the squares of its distances from the nearest points of the given circles shall be a minimum, the expression to minimize is

$$\Sigma(\left|\sqrt{(x - x_i)^2 + (y - y_i)^2}\right| - r_i)^2.$$

If we make a minute change of notation whereby

$$(8) \qquad \left|\sqrt{(x - x_i)^2 + (y - y_i)^2}\right| = d_i + r_i; \qquad \Sigma\frac{x_i}{r_i^2} = \Sigma\frac{y_i}{r_i^2} = 0$$

and equate to zero the partial derivatives with respect to x and y, we find

$$(9) \qquad \begin{aligned} \Sigma\frac{(x - x_i)d_i}{d_i + r_i} &= 0, \\ \Sigma\frac{(y - y_i)d_i}{d_i + r_i} &= 0. \end{aligned}$$

THEOREM. *The point sought is characterized by the fact that the geometric sum of the vectors from there to the nearest points of the given circles is zero.*

Let us look for an approximate solution, following our previous work step by step we reach

$$(10) \qquad \begin{aligned} \Sigma\frac{(x - x_i)^3}{r_i^2} + \Sigma\frac{(x - x_i)(y - y_i)^2}{r_i^2} - \Sigma(x - x_i) &= 0, \\ \Sigma\frac{(x - x_i)^2(y - y_i)}{r_i^2} + \Sigma\frac{(y - y_i)^3}{r_i^2} - \Sigma(y - y_i) &= 0. \end{aligned}$$

It would be pleasant to follow copy from here right to the end; unfortunately we can not do so, owing to the fact that we have nothing to correspond to the first equation (5). We may, however, rewrite (10) in the form

$$\Sigma[(x^2 + y^2) - 2(xx_i + yy_i) + (x_i^2 + y_i^2 - r_i^2)]\left(\frac{x - x_i}{r_i^2}\right) = 0,$$

$$\Sigma[(x^2 + y^2) - 2(xx_i + yy_i) + (x_i^2 + y_i^2 - r_i^2)]\left(\frac{y - y_i}{r_i^2}\right) = 0.$$

If we put

$$x^2 + y^2 = z$$

and remember (8), we easily find equations of the type

$$x = \frac{Az + B}{Pz^2 + Qz + R}, \qquad y = \frac{Cz + D}{Pz^2 + Qz + R}.$$

Here z is a root of a quintic equation with known coefficients. At this point the geometer steps lightly aside, leaving the quintic in possession of the field.

AMERICAN EXPEDITIONARY FORCES,
September, 1918.

SINGULAR SOLUTIONS OF DIFFERENTIAL EQUATIONS OF THE SECOND ORDER.

BY E. M. COON AND R. L. GORDON.

In the process of finding the singular solutions of differential equations of the first order and also in finding the envelopes of families of curves with one parameter, some extraneous loci appear. Analogous ones appear in the theory of singular solutions of differential equations of higher order, and of osculating curves of families of curves with more than one parameter. In discussions concerning singular solutions of differential equations of higher order attention is usually called to a locus of cusps of the second kind, and to a singular solution of the second kind, which has no analogue for differential equations of the first order, but we have found no mention or illustrations of a locus of cusps of the first kind, or of a tac-locus, where non-consecutive curves have contact of higher order. In this paper we give a few examples to illustrate cases that may arise for differential equations of the second order. We also consider the theory of osculating curves of families of curves with two parameters, and illustrate the fact that besides an osculating curve, a cusp locus or a nodal locus may appear.

In the first example the differential equation has two singular solutions, which give the osculating curves with contact of the second order with the family of integrals. In the second example there is a singular solution, and a locus of cusps of the first kind with an infinite number of cusps at each point. In the third example there is a tac-locus with contact of the second order, a locus of cusps of the first kind with an infinite number of cusps at each point, and in connection with the osculating curves a nodal-locus with an infinite number of nodes at each point. In the fourth example there is a singular solution and a locus of cusps of the first kind with one cusp at each point.

Let

(1) $$F(x, y, y', y'') = 0$$

be a differential equation of the second order, where F is a polynomial in x, y, y', y'' which cannot be broken up into factors, and is of degree $n(n > 1)$ in y''. A singular solution of this equation is a solution which is not contained in the general solution. Such a solution, if it exists, must* satisfy $\varphi_1(x, y, y') = 0$, a differential equation of the first order

* Forsyth, Theory of differential equations, part II, vol. III, chapter 15.

obtained by forming the *resultant* of (1) and

$$\frac{\partial F}{\partial y''} = 0 \tag{2}$$

to eliminate y''. The general solution of $\varphi_1 = 0$ is a *singular solution of the first kind* of $F = 0$, if it is a solution of $F = 0$, and a singular solution of $\varphi_1 = 0$ which satisfies $F = 0$, is a *singular solution of the second kind* of $F = 0$. Let

$$\varphi_2(x, y, y') = 0$$

be the differential equation of first order obtained by the elimination of y'' between (1) and

$$\frac{\partial F}{\partial x} + y'\frac{\partial F}{\partial y} + y''\frac{\partial F}{\partial y'} = 0 \tag{3}$$

A necessary condition* for the existence of a singular solution of $F = 0$ is that (1), (2) and (3) have a common solution in y'', or a necessary condition that $y = s(x)$ be a singular solution of $F = 0$, is that $y = s(x)$ satisfy $\varphi_1 = 0$ and $\varphi_2 = 0$. A sufficient condition for the existence of a singular solution is that (1), (2) and (3) have a common solution in y'', when $(\partial F/\partial x) + y'(\partial F/\partial y)$ and $(\partial F/\partial y')$ are not also zero for that value of y''.

In addition to the singular solution, $\varphi_1(x, y, y') = 0$ may yield a locus of cusps of either the first or second kinds, a tac-locus on which two non-consecutive curves have contact of second order, or a particular solution. In addition to the singular solution, $\varphi_2(x, y, y'') = 0$ may contain a locus of points where $y''' = 0$, a tac-locus on which two non-consecutive curves have contact of second order, or a particular solution.

The integral curves of $\varphi_1(x, y, y') = 0$, which are singular solutions, and the integral curves $f(x, y, a, b) = 0$ of $F(x, y, y', y'') = 0$ have contact of second order, that is, the integral curves of $\varphi_1 = 0$ are osculating curves of the integral curves of $F = 0$.

Let $f(x, y, a, b) = 0$ be the equation of a two-parameter family of curves. There may exist a one-parameter family of curves which has contact of second order with the given family of curves. It can be shown, in a manner similar to the method of finding the envelope of a one parameter family of curves, that the osculating curves satisfy the differential equation of first order $\varphi(x, y, y') = 0$, found by eliminating a,b, and db/da from the following four equations

$$f(x, y, a, b) = 0,$$

* Goursat, E., Sur les solutions singulières des équations différentielles simultanées, pp. 362–366, American Journal of Mathematics, vol. XI, 1888–89.

$$\frac{\partial f}{\partial x} + y'\frac{\partial f}{\partial y} = 0,$$

$$\frac{\partial f}{\partial a} + b'\frac{\partial f}{\partial b} = 0,$$

$$\frac{\partial^2 f}{\partial x\partial a} + b'\frac{\partial^2 f}{\partial x\partial b} + y'\left(\frac{\partial^2 f}{\partial y\partial a} + b'\frac{\partial^2 f}{\partial y\partial b}\right) = 0.$$

Besides the osculating curves, $\varphi(x, y, y') = 0$ may contain a locus of cusps of either the first or second kinds, or a nodal locus.

Example 1.

$$F = y''^3 - 4xy'y'' + 8y'^2 = 0,$$

$$f = (y - b) - a\frac{(x - a)^3}{3} = 0,$$

$$\varphi_1 = y'^3\left(y' - \frac{4x^3}{27}\right) = 0,$$

$$\varphi_2 = y'^4\left(y' - \frac{4x^3}{27}\right) = 0,$$

$$\varphi = y'\left(y' - \frac{4x^3}{27}\right) = 0,$$

$y' = 0$ and $y' - 4x^3/27 = 0$ satisfy $\varphi_1 = 0$, $\varphi_2 = 0$, and $\varphi = 0$; their integrals $y = c_1$, and $y - x^4/27 = c_2$ are singular solutions of the first kind of $F = 0$, and $y = c_1$ is also a particular solution.

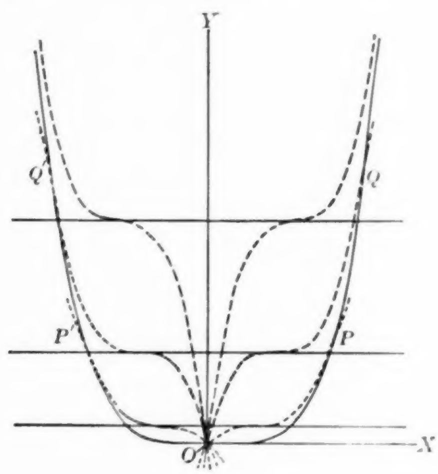

FIG. 1.

The dotted curves in Fig. 1 represent the integral curves, $f = 0$, obtained by giving the parameter a different values and the parameter

b the value zero. The equation of the curve $Q'P'OPQ$ is $y = x^4/27$; it is an osculating curve of the integral curves ($b = 0$), with contact of the second order. The straight lines $y = c_1$ are also osculating curves of the integral curves with contact of the second order at the points of inflection. Translation parallel to the Y-axis gives the complete system of the integral curves and their osculating curves.

Example 2.

$$F = 4xy''^2 + 2xy'' - y' = 0,$$

$$f = (y - b - ax) \pm \tfrac{2}{3}a^{1/2}x^{3/2} = 0,$$

$$\varphi_1 = x^2\left(y' + \frac{x}{4}\right) = 0,$$

$$\varphi_2 = y'\left(y' + \frac{x}{4}\right) = 0,$$

$$\varphi = x^3\left(y' + \frac{x}{4}\right) = 0.$$

$x = 0$ satisfies $\varphi_1 = 0$ and $\varphi = 0$, but does not satisfy $F = 0$; it is a cusp-locus. $y' + x/4 = 0$ satisfies $\varphi_1 = 0$, $\varphi_2 = 0$ and $\varphi = 0$, and the integrals $y + x^2/8 = c_1$ are singular solutions of $F = 0$. $y' = 0$ satisfies only $\varphi_2 = 0$, and the integrals $y_1 = c_2$ are particular solutions of $F = 0$.

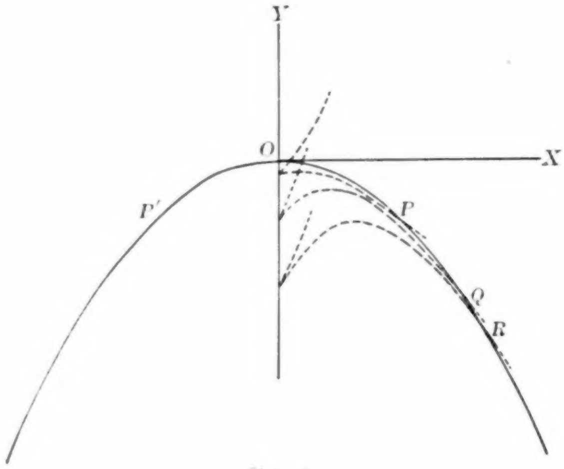

FIG. 2.

The dotted curves in Fig. 2 represent the integral curves, $f = 0$, obtained by giving the parameter a different values, and the parameter b the value zero. The equation of the curve $P'OPQR$ is $y + x^2/8 = 0$;

it is an osculating curve of the family ($b = 0$) with contact of the second order. The Y-axis is a locus of cusps of the first kind. Translation parallel to the Y-axis gives the complete system; there is an infinite number of cusps at each point of the Y-axis.

Example 3.

$$F = yy''^2 - y'^6(y-1)^2 = 0,$$

$$f = 3(x + ay - b) \pm \tfrac{4}{5}\, y^{3/2}(y - 5) = 0,$$

$$\varphi_1 = y^2 y'^6 (y-1)^2 = 0,$$

$$\varphi_2 = y'^{14}(y-1)^2[(y+1)^2 - 36y\, y'^2(y-1)^4] = 0,$$

$$\varphi = y^3(y-5)^2 = 0.$$

$y - 5 = 0$ satisfies only $\varphi = 0$; it is a nodal locus. $y - 1 = 0$ satisfies $\varphi_1 = 0$ and $\varphi_2 = 0$; it is a tac-locus. $y = 0$ satisfies $\varphi_1 = 0$ and $\varphi = 0$; it is a cusp-locus. $[(y+1)^2 - 36y\, y'^2(y-1)^4] = 0$ satisfies $\varphi_2 = 0$; it gives a locus of points where $y''' = 0$. $y' = 0$ satisfies $\varphi_1 = 0$ and $\varphi_2 = 0$; the integrals $y = c$ are particular solutions of $F = 0$.

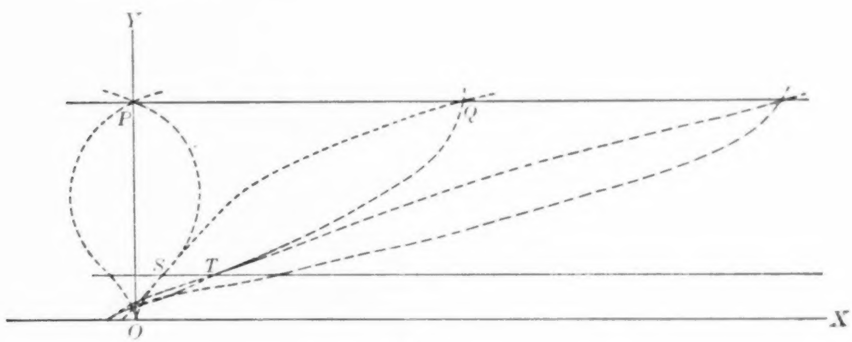

FIG. 3.

The dotted curves in Fig. 3 represent the integral curves, $f = 0$, obtained by giving the parameter a different values, and the parameter b the value zero. Translation parallel to the X-axis gives the complete system of integral curves. The equation of the straight line PQ is $y = 5$; there is an infinite number of nodes at each point. The equation of the straight line ST is $y = 1$; at each point there is an infinite number of pairs of non-consecutive curves with contact of the second order. At each point of the X-axis there is an infinite number of cusps of the first kind.

Example 4.

$$F = 8(y'' - 2)^2 y' - 9(y'' - 2) - 9 = 0,$$

$$f = y - b - (x - a)^2 \pm (x - a)^{3/2} = 0,$$

$$\varphi_1 = y'(32y' + 9) = 0,$$

$$\varphi_2 = (32y' + 9) = 0,$$

$$\varphi = y'^8(32y' + 9) = 0.$$

$y' = 0$ satisfies $\varphi_1 = 0$ and $\varphi_2 = 0$; the integrals $y = c_1$ are cusp-loci. $32y' + 9 = 0$ satisfies $\varphi_1 = 0$, $\varphi_2 = 0$, and $\varphi = 0$, and the integrals $y + 9x/32 = c_2$ are singular solutions of the first kind of $F = 0$.

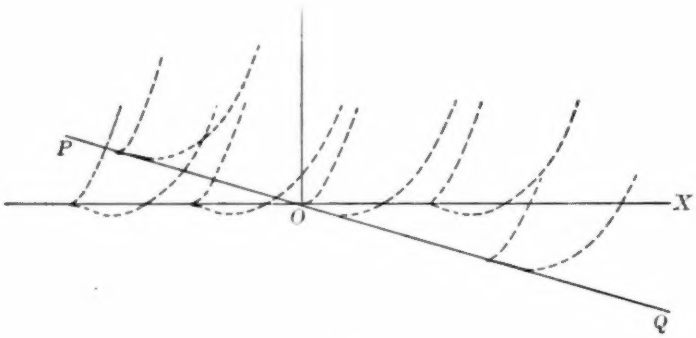

FIG. 4.

The dotted curves in Fig. IV represent the integral curves $f = 0$. The X-axis and lines parallel to it are loci of cusps of the first kind. The equation of the line PQ is $y + 9x/32 = 0$. The line PQ and lines parallel to it are osculating curves of the integral curves $f = 0$.

MT. HOLYOKE COLLEGE,
June 1918.

NOTE ON A CLASS OF INTEGRAL EQUATIONS OF THE SECOND KIND.

By Clyde E. Love.

§ 1. **Introduction.** The object of the present paper is to develop the elementary theory of the singular integral equation

$$(1) \qquad \varphi(x) = f(x) + \lambda \int_a^\infty K(x, t)\varphi(t)dt,$$

where $\varphi(x)$ is the unknown function, under the following assumptions:

The function $K(x, t)$ is bounded and integrable in the square

$$I_b: \quad a \leqq x \leqq b, \qquad a \leqq t \leqq b,$$

where b is arbitrary, and is continuous in the region

$$I: \quad x \geqq a, \qquad t \geqq a,$$

or if discontinuous has only a finite number of discontinuities for any one value of x or of t; the function $f(x)$ is continuous in the interval

$$I: \quad x \geqq a;$$

the functions $f(x)$ and $K(x, t)$ may be written in the respective forms

$$f(x) = f_1(x)/x^\alpha,$$

$$K(x, t) = K_1(x, t)/x^\alpha t^\beta,$$

where $f(x)$ and $K(x, t)$ are bounded in I and in T respectively, and where α and β are real constants such that $\alpha + \beta > 1$.

The singular integral equation has been studied by Weyl* and Hobson† under hypotheses differing but slightly from those assumed above. While the results of the present note thus possess only a modicum of novelty, it is felt that the case here treated is of some interest because, as will presently appear, it can be solved by a direct extension of the method of Fredholm.

In what follows we shall assume $a > 0$. This restriction involves no loss of generality, since it may be removed by a mere translation of axes.

It will be convenient to put $\alpha + \beta = \gamma + 1$, whence $\gamma > 0$.

* Math. Ann., Vol. 66 (1909), pp. 273–324.

† Proceedings of the London Math. Soc., Vol. 13 (1913), pp. 307–340.

§ 2. **Functions of the class** K. For brevity we shall say that, if a function satisfies all the conditions imposed above upon $K(x, t)$, that function *belongs to the class* K. We begin by noting certain properties of these functions.

First, if $K^{(1)}(x, t)$ is a function of the class K, the integral

$$\int_a^\infty |K^{(1)}(x, x)|dx$$

exists. For, the integral

$$\int_a^b K^{(1)}(x, x)dx$$

has a meaning for all values of $b > a$. Let $M^{(1)}$ denote the maximum value of $|K_1^{(1)}(x, t)|$ in the region T. Then, corresponding to every positive value of ϵ we can find a value of ξ such that

$$\int_\xi^{\xi'} |K^{(1)}(x, x)|dx < M^{(1)} \int_\xi^{\xi'} \frac{dx}{x^{\gamma+1}} < \frac{M^{(1)}}{\gamma\xi^\gamma} < \epsilon,$$

for all values of $\xi' > \xi$.

Further, if $K^{(2)}(x, t)$ is a second function of the class K, the integral

$$\bar{K}(x, t) = \int_a^\infty K^{(1)}(x, s)K^{(2)}(s, t)ds$$

converges absolutely for all values of x and t in the region T, represents a continuous function of x and t in T, and itself belongs to the class K.

To prove the first of these statements, we note that, the point (x, t) being any fixed point in T, the integral

$$\int_a^b K^{(1)}(x, s)K^{(2)}(s, t)ds$$

exists by hypothesis, and that to every ϵ there corresponds a ξ such that

$$\int_\xi^{\xi'} |K^{(1)}(x, s)K^{(2)}(s, t)|ds < \frac{M^{(1)}M^{(2)}}{x^\alpha t^\beta} \int_\xi^{\xi'} \frac{ds}{s^{\gamma+1}} < \frac{M^{(1)}M^{(2)}}{x^\alpha t^\beta} \cdot \frac{1}{\gamma\xi^\gamma} < \epsilon.$$

We shall prove the continuity of $\bar{K}(x, t)$ directly, by showing that to every positive ϵ there corresponds an η such that

$$(2) \qquad |\bar{K}(x', t') - \bar{K}(x, t)| < \epsilon$$

whenever

$$(3) \qquad |x - x'| < \eta, \qquad |t - t'| < \eta.$$

Having fixed ϵ, let us choose a number b satisfying the conditions

$$b > x, \qquad b > t,$$

$$b > \left(\frac{4M^{(1)}M^{(2)}}{a^{\gamma+1}\gamma\epsilon}\right)^{1/\gamma}. \tag{4}$$

Writing $\bar{K}(x, t)$ in the form

$$\bar{K}(x, t) = \int_a^b K^{(1)}(x, s)K^{(2)}(s, t)ds + \int_b^\infty K^{(1)}(x, s)K^{(2)}(s, t)ds,$$

we see that

$$|\bar{K}(x', t') - \bar{K}(x, t)| \leqq I_1 + I_2,$$

where

$$I_1 = \left|\int_a^b [K^{(1)}(x', s)K^{(2)}(s, t') - K^{(1)}(x, s)K^{(2)}(s, t)]ds\right|,$$

$$I_2 = \int_b^\infty \left[\frac{|K_1^{(1)}(x', s)K_1^{(2)}(s, t')|}{x'^\alpha t'^\beta} + \frac{|K_1^{(1)}(x, s)K_1^{(2)}(s, t)|}{x^\alpha t^\beta}\right]\frac{ds}{s^{\gamma+1}}.$$

Now it is well known* that the integral

$$\int_a^b K^{(1)}(x, s)K^{(2)}(s, t)ds$$

represents a continuous function of x and t in the region T_b; it follows at once that by suitable choice of η we may make

$$I_1 < \epsilon/2$$

for all values of x' and t' lying in T_b and satisfying the inequalities (3). As regards I_2, we have

$$I_2 \leqq M^{(1)}M^{(2)}\left[\frac{1}{x'^\alpha t'^\beta} + \frac{1}{x^\alpha t^\beta}\right]\int_b^\infty \frac{ds}{s^{\gamma+1}} < \frac{2M^{(1)}M^{(2)}}{a^{\gamma+1}\gamma b^\gamma} < \frac{\epsilon}{2},$$

by virtue of (4). Hence, if we place upon η the additional restrictions

$$\eta \leqq b - x, \qquad \eta \leqq b - t,$$

which is of course allowable, the relation (2) is established.

Finally, we note that

$$|\bar{K}(x, t)| \leqq \frac{M^{(1)}M^{(2)}}{x^\alpha t^\beta}\int_b^\infty \frac{ds}{s^{\gamma+1}} = \frac{M^{(1)}M^{(2)}}{x^\alpha t^\beta \gamma a^\gamma};$$

whence we may write

$$\bar{K}(x, t) = \bar{K}_1(x, t)/x^\alpha t^\beta,$$

where $\bar{K}_1(x, t)$ is bounded in T. This completes the proof that $\bar{K}(x, t)$ belongs to the class K.

* Cf. Heywood et Fréchet, L'équation de Fredholm, p. 6.

§ 3. **The method of Fredholm.** Let us put

$$K\begin{pmatrix} x_1, & x_2, & \cdots, & x_n \\ t_1, & t_2, & \cdots, & t_n \end{pmatrix} = |K(x_i, t_j)|, \qquad i, j = 1, 2, \cdots, n$$

$$= \frac{K_1\begin{pmatrix} x_1, & x_2, & \cdots, & x_n \\ t_1, & t_2, & \cdots, & t_n \end{pmatrix}}{(x_1x_2 \cdots x_n)^\alpha (t_1t_2 \cdots t_n)^\beta},$$

where

$$K_1\begin{pmatrix} x_1, & x_2, & \cdots, & x_n \\ t_1, & t_2, & \cdots, & t_n \end{pmatrix} = |K_1(x_i, t_j)|, \qquad i, j = 1, 2, \cdots, n;$$

also put

$$D(x, t, \lambda) = \sum_{n=0}^{\infty} u_n(x, t, \lambda), \tag{5}$$

$$D(\lambda) = \sum_{n=0}^{\infty} u_n(\lambda), \tag{6}$$

where

$$u_0(x, t, \lambda) = K(x, t),$$

$$u_n(x, t, \lambda) = \frac{(-\lambda)^n}{n!} \int_a^\infty \cdots \int_a^\infty K\begin{pmatrix} x, & x_1, & \cdots, & x_n \\ t, & x_1, & \cdots, & x_n \end{pmatrix} dx_1 \cdots dx_n,$$
$$(n = 1, 2, \cdots)$$

$$u_0(\lambda) = 1,$$

$$u_n(\lambda) = \frac{(-\lambda)^n}{n!} \int_a^\infty \cdots \int_a^\infty K\begin{pmatrix} x_1, & \cdots, & x_n \\ x_1, & \cdots, & x_n \end{pmatrix} dx_1 \cdots dx_n$$
$$(n = 1, 2, \cdots).$$

The convergence of the iterated integrals $u_n(x, t, \lambda)$, $u_n(\lambda)$ follows directly from the results of § 2. Further, denoting by M the maximum value of $|K_1(x, t)|$ in T, we have by Hadamard's theorem

$$\left| K_1\begin{pmatrix} x_1, & \cdots, & x_n \\ x_1, & \cdots, & x_n \end{pmatrix} \right| > \sqrt{n^n} M^n,$$

whence it follows very readily that series (6), and similarly (5), converges absolutely for all values of λ.

Thus in the present instance, just as in the case of the regular equation, the characteristic constants, or zeros of $D(\lambda)$, are isolated, and are determined by $K(x, t)$ alone. The "resolvent kernel"

$$K(x, t, \lambda) = D(x, t, \lambda)/D(\lambda) \tag{7}$$

is a meromorphic function of λ, and is defined as a function of x and t except when λ is a characteristic constant.

In what follows we confine our attention to the case where λ is not a characteristic constant.

§ 4. **Properties of the resolvent kernel.** It is easily seen that the function $K(x, t, \lambda)$ belongs to the class K. For, it follows directly from the convergence of series (5) that we may write

$$K(x, t, \lambda) = K_1(x, t, \lambda)/x^\alpha t^\beta \tag{8}$$

where $K_1(x, t, \lambda)$ is bounded in T. Further, the determinant

$$K\begin{pmatrix} x, & x_1, & \cdots, & x_n \\ t, & x_1, & \cdots, & x_n \end{pmatrix}$$

may evidently be written in the form

$$K\begin{pmatrix} x, & x_1, & \cdots, & x_n \\ t, & x_1, & \cdots, & x_n \end{pmatrix} = K(x, t)K\begin{pmatrix} x_1, & \cdots, & x_n \\ x_1, & \cdots, & x_n \end{pmatrix} + \Delta\begin{pmatrix} x, & x_1, & \cdots, & x_n \\ t, & x_1, & \cdots, & x_n \end{pmatrix},$$

where

$$\Delta\begin{pmatrix} x, & x_1, & \cdots, & x_n \\ t, & x_1, & \cdots, & x_n \end{pmatrix} = \begin{vmatrix} 0 & K(x, x_1) & \cdots & K(x, x_n) \\ K(x_1, t) & K(x_1, x_1) & \cdots & K(x_1, x_n) \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ K(x_n, t) & K(x_n, x_1) & \cdots & K(x_n, x_1) \end{vmatrix}.$$

Substituting in (5), we get

$$D(x, t, \lambda) = D(\lambda)K(x, t) + R(x, t, \lambda), \tag{9}$$

where

$$R(x, t, \lambda) = \sum_{n=1}^{\infty} \frac{(-\lambda)^n}{n!} \int_a^\infty \cdots \int_a^\infty \Delta\begin{pmatrix} x, & x_1, & \cdots, & x_n \\ t, & x_1, & \cdots, & x_n \end{pmatrix} dx_1 \cdots dx_n. \tag{10}$$

It follows from § 2 that each term of the series (10) is a continuous function of x and t, and it is readily seen that the series converges uniformly for all values of x and t in T; hence $R(x, t, \lambda)$ is continuous in T. By (9),

$$K(x, t, \lambda) = K(x, t) + R(x, t, \lambda)/D(\lambda).$$

Since $K(x, t, \lambda)$ is equal to the sum of $K(x, t)$ and a continuous function, and has the form (8), it belongs to the class K.

It is now easy to establish the following fundamental formula:

$$\begin{aligned} K(x, t, \lambda) - K(x, t) &= \lambda \int_a^\infty K(x, s)K(s, t, \lambda)ds \\ &= \lambda \int_a^\infty K(x, s, \lambda)K(s, t)ds. \end{aligned} \tag{11}$$

We note first that, after expanding $\Delta\begin{pmatrix} x, x_1, \cdots, x_n \\ t, x_1, \cdots, x_n \end{pmatrix}$ by minors of the first row, we may write*

$$\int_a^\infty \cdots \int_a^\infty \Delta\begin{pmatrix} x, x_1, \cdots, x_n \\ t, x_1, \cdots, x_n \end{pmatrix} dx_1 \cdots dx_n$$

$$= -n\int_a^\infty \cdots \int_a^\infty \int_a^\infty K(x, s)K(s, t)K\begin{pmatrix} x_1, \cdots, x_{n-1} \\ x_1, \cdots, x_{n-1} \end{pmatrix} dsdx_1 \cdots dx_{n-1}$$

$$- n\int_a^\infty \cdots \int_a^\infty \int_a^\infty K(x, s)\Delta\begin{pmatrix} s, x_1, \cdots, x_{n-1} \\ t, x_1, \cdots, x_{n-1} \end{pmatrix} dsdx_1 \cdots dx_{n-1}.$$

Hence

$$R(x, t, \lambda) = R_1 + R_2,$$

where

$$R_1 = \lambda \sum_{n=1}^{\infty} \frac{(-\lambda)^{n-1}}{(n-1)!} \int_a^\infty \cdots \int_a^\infty \int_a^\infty K(x, s)K(s, t)K\begin{pmatrix} x_1, \cdots, x_{n-1} \\ x_1, \cdots, x_{n-1} \end{pmatrix} dsdx_1 \cdots dx_{n-1}$$

$$= \lambda D(\lambda) \int_a^\infty K(x, s)K(s, t)ds,$$

and

$$R_2 = \lambda \sum_{n=1}^{\infty} \frac{(-\lambda)^{n-1}}{(n-1)!} \int_a^\infty \cdots \int_a^\infty \int_a^\infty K(x, s)\Delta\begin{pmatrix} s, x_1, \cdots, x_{n-1} \\ t, x_1, \cdots, x_{n-1} \end{pmatrix} dsdx_1 \cdots dx_{n-1}.$$

Let us set

$$f_n(s) = \frac{\lambda(-\lambda)^{n-1}}{n!} \int_a^\infty \cdots \int_a^\infty K(x, s)\Delta\begin{pmatrix} s, x_1, \cdots, x_{n-1} \\ t, x_1, \cdots, x_{n-1} \end{pmatrix} dx_1 \cdots dx_{n-1},$$

and apply the following theorem:†

If the series $\sum_{n=1}^{\infty} f_n(s)$ converges uniformly in any fixed interval (a, b), while the series

$$\sum_{n=1}^{\infty} \int_a^\infty f_n(s)ds$$

converges uniformly in an infinite interval $s \geqq a$, then we may write

$$\sum_{n=1}^{\infty} \int_a^\infty f_n(s)ds = \int^\infty \sum_{n=1}^{\infty} f_n(s)ds.$$

In the present instance, we have by Hadamard's theorem

$$|f_n(s)| < \frac{|\lambda|^n}{(n-1)!} \cdot \frac{\sqrt{n^n}M^{n+1}}{x^\alpha t^\beta s^{\gamma+1}} \int_a^\infty \cdots \int_a^\infty \frac{dx_1 \cdots dx_{n-1}}{(x_1 \cdots x_{n-1})^{\gamma+1}}$$

$$\leqq \frac{|\lambda|^n}{(n-1)!} \cdot \frac{\sqrt{n^n}M^{n+1}}{x^\alpha t^\beta a^{\gamma+1}} \cdot \frac{1}{\gamma^{n-1}a^{(n-1)\gamma}},$$

* Cf. Heywood et Fréchet, loc. cit., p. 55.

† Bromwich, Infinite Series, p. 455.

whence the series $\sum_{n=1}^{\infty} f_n(s)$ converges uniformly for all values of $s \geqq a$. Further, we have

$$\left| \int_a^{\infty} f_n(s)ds \right| < \frac{|\lambda|^n}{(n-1)!} \cdot \frac{\sqrt{n^n} M^{n+1}}{x^{\alpha} t^{\beta}} \cdot \frac{1}{\gamma^n a^{n\gamma}},$$

so that the second condition of the theorem is satisfied. Therefore we may write

$$R_2 = \lambda \int_a^{\infty} K(x, s) \left[\sum_{n=1}^{\infty} \frac{(-\lambda)^{n-1}}{(n-1)!} \int_a^{\infty} \cdots \int_a^{\infty} \Delta \begin{pmatrix} s, & x_1, & \cdots, & x_{n-1} \\ t, & x_1, & \cdots, & x_{n-1} \end{pmatrix} dx_1 \cdots dx_{n-1} \right] ds$$

$$= \lambda \int_a^{\infty} K(x, s) R(s, t, \lambda) ds,$$

or, combining (12) and (13),

$$R(x, t, \lambda) = \lambda D(\lambda) \int_a^{\infty} K(x, s) K(s, t) ds + \lambda \int_a^{\infty} K(x, s) R(s, t, \lambda) ds.$$

Substituting for $R(x, t, \lambda)$ its value from (9), we get

$$D(x, t, \lambda) - D(\lambda) K(x, t) = \lambda \int_a^{\infty} K(x, s) D(s, t, \lambda) ds,$$

or, dividing by $D(\lambda)$,

$$K(x, t, \lambda) - K(x, t) = \lambda \int_a^{\infty} K(x, s) K(s, t, \lambda) ds.$$

Expanding $\Delta \begin{pmatrix} x, & x_1, & \cdots, & x_n \\ t, & x_1, & \cdots, & x_n \end{pmatrix}$ by minors of the first column and proceeding similarly, we obtain the second desired formula,

$$K(x, t, \lambda) - K(x, t) = \lambda \int_a^{\infty} K(x, s, \lambda) K(s, t) ds.$$

§ 5. **The fundamental theorem.** We are now in position to prove the following

THEOREM: *If the functions $K(x, t)$ and $f(x)$ satisfy the conditions of* § 1, *and if λ is not a characteristic constant, the equation*

$$\varphi(x) = f(x) + \lambda \int_a^{\infty} K(x, t) \varphi(t) dt$$

has a unique continuous solution given by the formula

$$\varphi(x) = f(x) + \lambda \int_a^{\infty} K(x, t, \lambda) f(t) dt,$$

where $K(x, t, \lambda)$ is a meromorphic function of λ defined by equations (5), (6), (7). *This solution may be written in the form*

(14) $$\varphi(x) = \varphi_1(x)/x^{\alpha},$$

where $\varphi_1(x)$ is bounded in the interval $x \geqq a$.

First, assume that a solution of the form (14) exists.* Then

$$\varphi(t) = f(t) + \lambda \int_a^{\infty} K(t, t_1)\varphi(t_1)dt_1.$$

Multiplying by $K(x, t, \lambda)$ and integrating from $t = a$ to $t = \infty$, we find

$$\int_a^{\infty} K(x,t,\lambda)\varphi(t)dt = \int_a^{\infty} K(x,t,\lambda)f(t)dt + \lambda \int_a^{\infty}\int_a^{\infty} K(x,t,\lambda)K(t,t_1)\varphi(t_1)dt_1dt;$$

it is easily shown that all the integrals here occurring exist. By means of (11), this equation is transformed into

$$\int_a^{\infty} K(x, t, \lambda)\varphi(t)dt = \int_a^{\infty} K(x, t, \lambda)f(t)dt + \int_a^{\infty} [K(x, t, \lambda) - K(x, t)]\varphi(t)dt,$$

whence

$$\int_a^{\infty} K(x, t, \lambda)f(t)dt = \int_a^{\infty} K(x, t)\varphi(t)dt.$$

Substituting in (1) we get

(15) $$\varphi(x) = f(x) + \lambda \int_a^{\infty} K(x, t, \lambda)f(t)dt.$$

It is evident at once that the function $\varphi(x)$ defined by (15) may be written in the form (14). For, let M_1 and M_2 denote the maximum values of $|K_1(x, t, \lambda)|$ and $|f_1(x)|$ respectively. Then

$$|\varphi(x)| \leqq \frac{M_2}{x^{\alpha}} + \frac{|\lambda| M_1 M_2}{x^{\alpha}} \int_a^{\infty} \frac{dt}{t^{\gamma+1}} = \frac{1}{x^{\alpha}}\left[M_2 + \frac{|\lambda| M_1 M_2}{\gamma a^{\gamma}} \right].$$

To see that the assumed solution exists, let us consider (15) as an integral equation in $f(x)$. The above argument, when applied to this equation, shows that the solution $f(x)$ is given by the formula

$$f(x) = \varphi(x) - \lambda \int_a^{\infty} K(x, t)\varphi(t)dt,$$

so that $\varphi(x)$ as given by (15) is a solution of (1).

Finally, assume that a second solution $\varphi_1(x)$ exists, so that

(16) $$\varphi_1(x) = f(x) + \lambda \int_a^{\infty} K(x, t)\varphi_1(t)dt.$$

* Cf. Heywood et Fréchet, loc. cit., p. 40.

Eliminating $f(x)$ between (15) and (16), we find

$$\varphi(x)-\varphi_1(x)=\lambda\int_a^\infty \varphi_1(t)\left[K(x,t,\lambda)-K(x,t)-\lambda\int_a^\infty K(x,s,\lambda)K(s,t)ds\right]dt,$$

or, by (11),

$$\varphi(x) - \varphi_1(x) = 0.$$

This completes the proof of the theorem.

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CONCERNING SENSE ON CLOSED CURVES IN NON-METRICAL PLANE ANALYSIS SITUS.*

By John Robert Kline.

§ 1. **Introduction.** In a paper recently published in these Annals,† I proposed the following non-intuitional definition‡ for sameness of sense on closed curves in non-metrical plane analysis situs: *The sense $A_1B_1C_1$§ on the closed curve J_1 is said to be the same as the sense $A_2B_2C_2$ on the closed curve J_2 with respect to E_{12},‖ if there exists in E_{12} a closed curve J_3 and three points A_3, B_3 and C_3 thereon such that* (1) *if it is impossible to join A_1 to A_3, B_1 to B_3 and C_1 to C_3 by arcs, no two of which have a point in common and which lie except for their endpoints in E_{13}, then it is also impossible to join A_2 to A_3, B_2 to B_3 and C_2 to C_3 by arcs, no two of which have a point in common and which lie except for their endpoints in E_{23}* (2) *if it is possible to join A_1 to A_3, B_1 to B_3 and C_1 to C_3 as indicated above, then it is also possible to join A_2 to A_3, B_2 to B_3 and C_2 to C_3 as indicated. Otherwise the sense $A_1B_1C_1$ on J_1 and the sense $A_2B_2C_2$ on J_2 are said to be opposite with respect to E_{12}.*

A question might naturally arise whether the sense thus defined coincides with "the ordinary sense." The object of the present paper is: (1) To give a set of three independent postulates or axioms in terms of the undefined symbol "sameness of sense"; (2) to show that, if Σ' is any definition for sameness of sense on plane closed curves, then a necessary and sufficient condition that Σ' should have the properties of Axioms 1 to 3, is that Σ' be equivalent to Σ.

Consider the following axioms for sameness of sense on closed curves in a space satisfying Professor R. L. Moore's system of axioms Σ_3¶:

* Presented to the American Mathematical Society, April 26, 1919.

† Cf. A Definition of Sense on Closed Curves in non-metrical Plane Analysis Situs, these Annals, 2d series, vol. 19 (1918), pp. 185–200. Hereafter in the present paper, the above paper will be referred to as Definition.

‡ This definition will be referred to as Definition Σ.

§ In the present paper, when speaking of the sense BCA on the closed curve J, we shall understand that A, B and C are distinct points of J.

‖ Hereafter in this paper E_i will denote the exterior of the closed curve J_i while I_i denotes its interior. The symbol E_{ik} will denote the common exterior of J_i and J_k.

¶ Cf. R. L. Moore, "On the foundations of plane analysis situs," Transactions of the American Mathematical Society, vol. 17 (1916), pp. 131–64. It should be remembered that Σ_3 is satisfied even by certain spaces which are neither metrical, descriptive or separable.

AXIOM 1. *If A and C separate B and D on the closed curve J_1, then the sense ABC on J_1 is not the same as the sense ADC on J_1.**

AXIOM 2. *If, on the arc ABC of the closed curve J, E, H and F are points such that, on the arc ABC, the order $AEHFC$ holds while EXF is an arc such that $\underset{\smile}{EXF}$† is a subset of I, then the sense ABC on J is the same as the sense EHF on $EHFXE$.*

AXIOM 3. *If the sense $A_1B_1C_1$ on J_1 is both the same as the sense $A_2B_2C_2$ on J_2 and the same as the sense $A_3B_3C_3$ on J_3, then the sense $A_2B_2C_2$ on J_2 is the same as the sense $A_3B_3C_3$ on J_3.*

§ 2. **Consequences of Axioms 1–3.** THEOREM 1: *If A and B separate C and D on the closed curve J_1 while A_1 and B_1 are points of the arc ACB such that the order AA_1CB_1B holds on ACB, then the sense ACB on J_1 is the same as the sense A_1CB_1 on J_1.*

PROOF. Let F and G denote points of the arc ACB such that the order AA_1FCGB_1B holds, while FXG is an arc such that $\underset{\smile}{FXG}$ is a subset of I_1. Then, by Axiom 2, the sense FCG on $FCGXF$ is the same as the sense ACB on J_1 and is also the same as the sense A_1CB_1 on J_1. Hence, by Axiom 3, the sense ACB on J_1 is the same as the sense A_1CB_1 on J_1.

THEOREM 2. *If A and C separate B and D on the closed curve J_1, while AXC is an arc such that $\underset{\smile}{AXC}$ is a subset of I_1, then the sense ABC on $ABCXA$ is the same as the sense ABC on J.*

PROOF. Let E and H be points of J_1 such that the order $ABCHDEA$ holds. By Theorem 1, the sense EBH on J_1 is the same as the sense ABC on J_1. By Axiom 2, the sense EBH on J_1 is the same as the sense ABC on $ABCXA$. Hence, by Axiom 3, the sense ABC on J_1 is the same as the sense ABC on $ABCXA$.

THEOREM 3. *If A and C separate B and D on the closed curve J, then the sense ABC on J is the same as both the sense BCD on J and CDA on J.*

PROOF. Let M, O, and N be points of J such that the order $ABMONCDA$ holds while MEN is an arc such that $\underset{\smile}{MEN}$ is a subset of I.

* If A, B and C are distinct points of the closed curve J, we are at liberty to regard the sense ABC with respect to the interior of J or with respect to the exterior of J. However, in the present paper, we shall regard it with respect to the exterior of J although there would be no change in any of our results were we to regard sense always with respect to the interior of J. Let it be further understood in the present paper that the statement "the sense $A_1B_1C_1$ on J_1 is the same as the sense $A_2B_2C_2$ on J_2" is an abbreviation for the following statement, "The sense $A_1B_1C_1$ on J_1 with respect to E_1, the exterior of J_1, is the same as the sense $A_2B_2C_2$ on J_2 with respect to E_2, the exterior of J_2." It can also be seen that we may conveniently adopt the same convention in Σ where we shall replace the statement "the sense $A_1B_1C_1$ on J_1 is the same as the sense $A_2B_2C_2$ on J_2 with respect to E_{12}, the common exterior of J_1 and J_2" by the statement "the sense $A_1B_1C_1$ on J_1 is the same as the sense $A_2B_2C_2$ on J_2" interpreted as above.

† If ABC is an arc, then the symbol $\underset{\smile}{ABC}$ denotes the point set $ABC - A - B$.

It is clear that on the arc ABC the order $AMONC$ holds while on the arc BCD the order $BMOND$ holds. By Axiom 2 the sense ABC on J is the same as the sense MON on $MONEM$. By Axiom 2, the sense BCD on J is the same as the sense MON on $MONEM$. It follows, by Axiom 3, that the sense ABC on J is the same as the sense BCD on J.

In like manner, the sense BCD on J is the same as the sense CDA on J. Hence by Axiom 3, the sense ABC on J is the same as the sense CDA on J.

Theorem 4. *If the sense $A_1B_1C_1$ on J_1 is the same as the sense $A_2B_2C_2$ on J_2 with respect to E_{12} but is not the same as the sense $A_3B_3C_3$ on J_3, then the sense $A_2B_2C_2$ is not the same as the sense $A_3B_3C_3$ on J_3.*

The truth of Theorem 4 is evident at once from Axiom 3 by a "reductio ad absurdum" argument.

Theorem 5. *If A and C separate D and E on the closed curve J_1, while ABC is an arc such that $\underset{\smile}{ABC}$ is a subset of I_1, then the sense ABC on $ABCDA$ is not the same as the sense ABC on $ABCEA$.*

Proof. By Theorem 3, the sense ABC on $ABCDA$ and the sense ABC on $ABCEA$ are, respectively, the same as the sense CDA on $ABCDA$ and the sense CEA on $ABCEA$. By Theorem 2, the sense CEA on $ABCEA$ is the same as the sense CEA on J and the sense CDA on $ABCDA$ is the same as the sense CDA on J. It follows, by Axiom 3, that the sense ABC on $ABCDA$ is the same as the sense CDA on J while the sense ABC on $ABCEA$ is the same as the sense CEA on J. Let us suppose Theorem 5 false. Then the sense ABC on $ABCDA$ is the same as the sense ABC on $ABCEA$. It would follow, with the aid of Axiom 3, that the sense CDA on J was the same as the sense CEA on J. But this contradicts Axiom 1. Thus the supposition that our theorem is false, has led to a contradiction.

Theorem 6. *If A and C separate D and E on the closed curve J while ABC is an arc such that $\underset{\smile}{ABC}$ is a subset of I, then the sense ADC on J is the same as the sense ABC on $ABCEA$.*

Proof. By Theorem 3, the sense ADC on J is the same as the sense CEA on J. By Theorem 2, the sense CEA on $ABCEA$ is the same as the sense CEA on J. Hence, by Axiom 3, the sense ADC on J is the same as the sense CEA on $ABCEA$. By Theorem 3, the sense ABC on $ABCEA$ is the same as the sense CEA on $ABCEA$. It follows, by Axiom 3, that the sense ADC on J is the same as the sense ABC on $ABCEA$.

Theorem 7. *Suppose J_1 and J_2 are two simple closed curves such that $J_1 + I_1$ is a subset of E_2. Then a necessary and sufficient condition that*

the sense $A_1B_1C_1$ *on* J_1 *be not the same as the sense* $A_2B_2C_2$ *on* J_2, *is that it be possible to join* $A_1B_1C_1$ *to* $A_2B_2C_2$ *simply*.*

PROOF. (*a*) *The condition is necessary.* There exists at least one pair of arcs A_1XA_2 and C_1ZC_2, which have no point in common and lie except for their endpoints in E_{12}. Suppose the condition were not necessary. Then, for every pair of arcs, A_1XA_2 and C_1ZC_2 described above either (1) I_1 is within and I_2 is without or (2) I_1 is without and I_2 is within $A_1B_1C_1ZC_2B_2A_2XA_1$.†

CASE I. Suppose I_1 is within and I_2 without $A_1B_1C_1ZC_2B_2A_2XA_1$. Let D_i ($i = 1, 2$) be a point of the closed curve J_i such that A_i and C_i separate B_i and D_i on J_i. It may easily be proved that the interior of $A_1B_1C_1ZC_2D_2A_2XA_1 = A_1D_1C_1 + A_2B_2C_2 + I_1 + I_2 +$ the interior of $A_1D_1C_1ZC_2B_2A_2XA_1$. By Theorem 1, the sense $A_1B_1C_1$ on $A_1B_1C_1$-$ZC_2D_2A_2XA_1$ is the same as the sense $A_2B_1C_2$ on $A_1B_1C_1ZC_2D_2A_2XA_1$. By Theorem 2, the sense $A_1B_1C_1$ on J_1 is the same as the sense $A_1B_1C_1$ on $A_1B_1C_1ZC_2D_2A_2XA_1$. Hence, by Axiom 3, the sense $A_1B_1C_1$ on J_1 is the same as the sense $A_2B_1C_2$ on $A_1B_1C_1ZC_2D_2A_2XA_1$. By Theorem 6, the sense $A_2B_1C_2$ on $A_1B_1C_1ZC_2D_2A_2XA_1$ is the same as the sense $A_2B_2C_2$ on J_2. Hence, by Axiom 3, the sense $A_1B_1C_1$ on J_1 is the same as the sense $A_2B_2C_2$ on J_2. Thus, in Case I, we are led to a contradiction, if we suppose the condition not necessary.

CASE II. I_2 is within and I_1 without $A_1B_1C_1ZC_2B_2A_2XA_1$. In Case II, we are led to a contradiction if we suppose the condition is not necessary, just as in Case I.

(*b*) *The condition is sufficient.* Suppose $A_1B_1C_1$ and $A_2B_2C_2$ can be simply joined. Consider one set of arcs A_1XA_2, B_1YB_2 and C_1ZC_2 no two of which have a point in common and which lie, except for their endpoints entirely in E_{12}. Then I_1 and I_2 are either both within or both without $A_1B_1C_1ZC_2B_2A_2XA_1$.‡

CASE I*b*. I_1 and I_2 are both without $A_1B_1C_1ZC_2B_2A_2XA_1$. Let D_i [$i = 1, 2$] denote a point of J_i such that A_i and C_i separate B_i and D_i on J_i. It may be easily proved that I_1 is within and I_2 is without $A_1D_1C_1ZC_2B_2A_2XA_1$. Hence, by an argument similar in all respects to that given in Case I*a*, it may be proved that the sense $A_1D_1C_1$ on J_1 is the same as the sense $A_2B_2C_2$ on J_2. By Axiom 1, the sense $A_1D_1C_1$ on J_1

* If A_i, B_i and C_i are distinct points of the closed curve J_i ($i = 1, 2$) where J_1 and J_2 are closed curves such that $J_1 + I_1$ is a subset of E_2, then we shall say that $A_1B_1C_1$ can be *simply joined to* $A_2B_2C_2$, if and only if, there exist arcs A_1XA_2, B_1YB_2 and C_1ZC_2, no two of which have a point in common and which lie except for their end points in E_{12}.

† Cf. Definition, Theorems *B* and *H*, loc. cit., pp. 187 and 195.

‡ Cf. Definition, Theorems *B* and *H*, loc. cit., pp. 187–8.

is not the same as the sense $A_2B_2C_2$ on J_2. By Theorem 4, the sense $A_1B_1C_1$ on J_1 is not the same as the sense $A_1B_1C_1$ on J_1.

CASE II*b*. I_1 and I_2 are both within $A_1B_1C_1ZC_2B_2A_2XA_1$. The proof is similar to the proof in Case I*b*.

THEOREM 8. *In order that the sense $A_1B_1C_1$ on the closed curve J_1 be the same as the sense $A_2B_2C_2$ on J_2, it is both necessary and sufficient that there exist in E_{12}, a closed curve J_3 and three points A_3, B_3 and C_3 thereon such that $A_1B_1C_1$ and $A_2B_2C_2$ can both be simply joined to $A_3B_3C_3$.*

PROOF. (*a*) *The condition is necessary.* Let J_3 denote any closed curve such that $J_i + I_i$ $(i = 1, 2)$ is a subset of E_3. On J_3 select four points A_3', B_3', C_3' and D_3' such that A_3' and C_3' separate B_3' and D_3' on J_3. Two cases may arise:

CASE I*a*. $A_1B_1C_1$ and $A_3'B_3'C_3'$ can be simply joined. Then $C_3'B_3'A_3'$ and $A_1B_1C_1$ cannot be simply joined.* Hence, as in the proof of Theorem 7, the sense $C_3'B_3'A_3'$ on J_3 is the same as the sense $A_1B_1C_1$ on J_1. Then it must also be impossible to join $C_3'B_3'A_3'$ and $A_2B_2C_2$ simply. For, if it were possible to join $C_3'B_3'A_3'$ and $A_2B_2C_2$ simply, then by Theorem 7, the sense $A_2B_2C_2$ on J_2 would not be the same as the sense $A_3B_3C_3$ on J_3, which in turn would imply, by Theorem 4, that the sense $A_1B_1C_1$ on J_1 was not the same as the $A_2B_2C_2$ on J_2. But this is contrary to hypothesis. Hence $C_3'B_3'A_3'$ and $A_2B_2C_2$ cannot be simply joined. Then $A_2B_2C_2$ and $A_3'B_3'C_3'$ can be simply joined.†

CASE II*a*. $A_1B_1C_1$ and $A_3'B_3'C_3'$ cannot be simply joined. Then it follows as in Case I*a*, that $A_2B_2C_2$ and $A_3'B_3'C_3'$ cannot be simply joined. But then $C_3'B_3'A_3'$ can be simply joined to both $A_1B_1C_1$ and $A_2B_2C_2$.

In Case I(*a*) let $A_3B_3C_3$ denote $A_3'B_3'C_3'$ while in Case II*a* $A_3B_3C_3$ denotes $C_3'B_3'A_3'$.

(*b*) *The condition is sufficient.* Suppose there exists in E_{12} a closed curve J_3 and three points A_3, B_3 and C_3 thereon such that $A_3B_3C_3$ can be simply joined to both $A_1B_1C_1$ and $A_2B_2C_2$. It follows from the definition of simple joining that $J_i + I_i$ $(i = 1, 2)$ must lie entirely in E_3. As $A_3B_3C_3$ can be simply joined to both $A_1B_1C_1$ and $A_2B_2C_2$, it follows that $C_3B_3A_3$ can be simply joined to neither $A_1B_1C_1$ nor $A_2B_2C_2$. Then, by Theorem 7, the sense $C_3B_3A_3$ on J_3 is the same as the sense $A_1B_1C_1$ on J_1 and is also the same as the sense $A_2B_2C_2$ on J_2. Hence, by Axiom 3, the sense $A_1B_1C_1$ on J_1 is the same as the sense $A_2B_2C_2$ on J_2.

THEOREM 9. *Suppose Σ' is any definition of sense in a space satisfying Σ_3. Then a necessary and sufficient condition that Σ' have the properties of Axioms 1 to 3, is that Σ' be equivalent to Σ.*

* Cf. Definition, loc. cit., Theorem *K*, pp. 197–8.

† Cf. Definition, Theorem *K*, loc. cit., pp. 197–8.

PROOF. (*a*) *The condition is necessary.* If Axioms 1–3 are satisfied, then Theorems 7 and 8 of the present paper hold and the sense $A_1B_1C_1$ on J_1 is the same as the sense $A_2B_2C_2$ on J_2 if and only if, there exists in E_{12} a closed curve J_3 and three points A_3, B_3 and C_3 thereon such that $A_3B_3C_3$ can be simply joined to both $A_1B_1C_1$ and $A_2B_2C_2$.

(*b*) *The condition is sufficient.* Suppose Σ'', any definition of sense in Σ_3, is equivalent to Σ. We shall now show that Axioms 1 to 3 are satisfied.

(1) *Axiom 1 is satisfied.* Suppose the points A_1 and C_1 separate B_1 and D_1 on J_1. Choose a simple closed curve J_3 and three points A_3, B_3, and C_3 thereon such that $A_1B_1C_1$ and $A_3B_3C_3$ can be simply joined and such that $J_1 + I_1$ is a subset of E_3. Then let A_1ZA_3 and C_1XC_3 be any two arcs having no point in common and lying except for their end-points in E_{13}. Then either I_1 and I_3 are both within or I_1 and I_3 are both without* $A_1B_1C_1XC_3B_3A_3XA_1$. It may easily be proved that either (1) I_1 is without and I_2 within or (2) I_1 is within and I_2 without $A_1D_1C_1X$-$C_3B_3A_3XA_1$. But then $A_3B_3C_3$ and $A_1D_1C_1$ cannot be simply joined. Hence, by Definition, the sense $A_1B_1C_1$ on J_1 is not the same as the sense $A_1D_1C_1$ on J_1.

(2) *Axiom 2 is satisfied.* Suppose (1) that the points A_1 and C_1 separate B_1 and D_1 on the closed curve J_1, (2) M, O and N are points of the arc $A_1B_1C_1$ in the order A_1MONC_1, (3) MQN is an arc such that $\overparen{MQN}$ is a subset of I_1. Consider the closed curve J_3 and the three points A_3, B_3 and C_3 thereon described in the above proof that Axiom 1 is satisfied. Let $\overline{A_1M}$ and $\overline{C_1N}$ denote, respectively the arcs of J_1 from A_1 to M and from C_1 to N which fail to contain D_1. Let A_3RM denote the arc $A_1ZA_3 + \overline{A_1M}$ while C_3SN denotes the arc $C_1XC_3 + \overline{C_1N}$. It follows that either (1) I_3 and the interior of $MONQM$ are both within or (2) the interior of both of these closed curves is without $A_3B_3C_3SNOMRA_3$. Hence $A_3B_3C_3$ and MON can be simply joined. It follows, by Definition, that the sense $A_1B_1C_1$ on J_1 is the same as the sense MON on $MONQM$.

(3) *Axiom 3 is satisfied.* See Theorem 2 of Definition.†

§ 3. **Questions of Independence.** In the following E_i is a definition of sense on closed curves in Σ_3 such that (1) Axiom i is not satisfied, (2) all other axioms of the set 1–3 are satisfied.

E_1. The sense $A_1B_1C_1$ on J_1 is the same as the sense $A_2B_2C_2$ on J_2 for every choice of $A_1B_1C_1$, $A_2B_2C_2$, J_1 and J_2.

* Cf. Definition, Theorems *B* and *H*, loc. cit., pp. 187 and 195.

† Cf. loc. cit., p. 199.

E_2. The sense $A_1B_1C_1$ on J_1 is the same as the sense $A_2B_2C_2$ on J_2 if and only if (1) J_1 and J_2 are identical, (2) A_1, B_1 and C_1 are respectively, identical with A_2, B_2 and C_2.

E_3. The sense $A_1B_1C_1$ on J_1 is the same as the sense $A_2B_2C_2$ on J_2, if and only if (1) the senses are the same according to Σ and (2) J_1 and J_2 have at least one common point.

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ON THE THEORY OF SUMMABILITY.*

By Glenn James.

Silverman† has defined the sum of a divergent series, whose sum to i terms is s_i, to be

(I) $$\lim_{x\to\infty}\lim_{n\to\infty}\sum_1^n s_i a_i(x)$$

where, (1) $\lim\limits_{x\to\infty} a_i(x) = 0$, (2) $\sum\limits_1^n |a_i(x)| < K$ or $a_i(x) \equiv 0$, and (3) $\lim\limits_{n=\infty}\sum\limits_1^n a_i(x) = 1$; and he proved that this definition is regular.‡ This paper considers similar but more general summation processes. A theorem which is fundamental in the theory of summability is developed. From this theorem certain deductions are made. The most important of these are: A test for the equivalence§ of two summation definitions, the regularity of the most|| general definition of the type of (I) and a theorem on the sum of certain product series. Incidentally it is proved that no regular definition which employs positive convergence factors can sum properly divergent series. In the latter part of the paper these theorems are transformed so as to apply to definitions based directly upon the *terms* of the series to be summed.

Consider the repeated limit, $\lim\limits_{x\to\infty}\lim\limits_{n\to\infty}\sum\limits_1^n \phi_i(n, x)\alpha_i(n, x)$, the functions $\phi_i(n, x)$ and $\alpha_i(n, x)$ being defined for; i any positive integer, n a positive integer equal to or greater than i, and x any real number. We have the following fundamental theorem.

Theorem I. *Given*:

(a) $$\lim_{x\to\infty}\lim_{n\to\infty}\alpha_i(n, x) = 0,$$

i fixed,

(b) $$\sum_1^n |\alpha_i(n, x)| < K,$$

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* Presented to the American Mathematical Society, under a different title, December, 1916.

† Silverman, University of Missouri Studies, Vol. 1, No. 1. For other general definitions of summability see: Chapman, Quarterly Journal of Mathematics, Vol. 43, and Smail, Dissertation, Columbia University.

‡ A definition is said to be *regular* when it sums every convergent series to its ordinary sum.

§ Two definitions are said to be *equivalent* when every series summed by one is summed by the other to the same sum.

|| "The most general" in the sense that its assumptions, as will be seen, are necessary as well as sufficient to establish its regularity.

K properly chosen,

(c) $$|\phi_i(n, x)| \leqq e_i,$$

for all values of x and n, where e_i, $i = 1, 2, 3, \cdots$, is a sequence of constants with limit zero. It follows from these assumptions that

$$\lim_{x\to\infty}\lim_{n\to\infty}\sum_1^n \phi_i(n, x)\alpha_i(n, x) = 0.$$

PROOF. Evidently we have

(1) $$\left|\sum_1^n \phi_i(n, x)\alpha_i(n, x)\right| \leqq \left|\sum_1^I \phi_i(n, x)\alpha_i(n, x)\right| + \left|\sum_{I+1}^n \phi_i(n, x)\alpha_i(n, x)\right|$$

(2) $$\leqq e\left|\sum_1^I \alpha_i(n, x)\right| + \sum_{I+1}^n |\phi_i(n, x)|\cdot|\alpha_i(n, x)|,$$

where $e \geqq e_i$, $i = 1, 2, 3, \cdots$. Now for an arbitrary ϵ we can choose an I such that

(3) $$|\phi_i(n, x)| \leqq e_i < \epsilon/2K,$$

for every $i > I$. Then with I fixed we can, because of assumption (a), choose X and N_x, (N depending on x), such that

(4) $$\left|\sum_1^I \alpha_i(n, x)\right| < \epsilon/2e$$

for $x > X$ and $n > N_x$. Applying (3) and assumption (b) to the second term and (4) to the first term in the right member of (2), we obtain

$$\left|\sum_1^n \phi_i(n, x)\alpha_i(n, x)\right| < e(\epsilon/2e) + K(\epsilon/2K) = \epsilon,\ x > X \text{ and } n > N_x,$$

from which it follows that

$$\lim_{x\to\infty}\lim_{n\to\infty}\sum_1^n \phi_i(n, x)\alpha_i(n, x) = 0.$$

As a corollary of this theorem we have the following test for the equivalence of two definitions of the sum of a series. Let

(A) $$S = \lim_{x\to\infty}\lim_{n\to\infty}\sum_1^n s_i\beta_i(n, x)$$

and

(B) $$S = \lim_{x\to\infty}\lim_{n\to\infty}\sum_1^n s_i\beta_i'(n, x)$$

be two such definitions, where s_i is the ordinary sum of the series to i terms.

COROLLARY I. *If (A) sums a series, (B) sums it to the same sum provided the difference, $(s_i\beta_i' - s_i\beta_i)$ can be expressed as the product of two*

factors α_i and ϕ_i such that α_i satisfies (a) and (b) and ϕ_i satisfies (c) of Theorem I.

For we can write

$$\sum_1^n s_i\beta_i' = \sum_1^n (s_i\beta_i + s_i\beta_i' - s_i\beta_i) = \sum_1^n s_i\beta_i + \sum_1^n (s_i\beta_i' - s_i\beta_i)$$

and the repeated limit of the second term on the right is zero by hypothesis.

We will apply this test to show the equivalence of the definitions,

(1) $$S = \lim_{n\to\infty} 1/n \sum_1^n s_i\text{*}$$

and

(2) $$S = \lim_{n\to\infty} \sum_1^n s_i \frac{i+k}{i(n+k)}.$$

Here

$$s_i\beta_i' - s_i\beta_i = \pm \frac{s_i(ik-nk)}{ni(n+k)},$$

the sign depending upon whether we set $1/n = \beta_i'$, or $1/n = \beta_i$. Evidently a necessary condition for the existence of either (1) or (2) is

$$\lim_{n\to\infty} s_n/n = 0.$$

From this condition and the fact that $n \geqq i$ it follows that

$$\lim_{i\to\infty} \frac{s_i(ik-nk)}{i(n+k)} = 0.$$

Consequently, if (2) is known to exist, we choose

$$\frac{s_i(ik-sk)}{i(n+k)} = \phi_i \qquad \text{and} \qquad 1/n = \alpha_i,$$

while if (1) is known to exist, we choose

$$-s_i/i = \phi_i \qquad \text{and} \qquad \frac{ik-nk}{n(n+k)} = \alpha_i.$$

These forms of α_i and ϕ_i satisfy the assumptions of the corollary.

COROLLARY II. *Given:*

(*a*) $$\lim_{x\to\infty}\lim_{n\to\infty} \alpha_i(n, x) = 0,$$

(*b*) $$\sum_1^n |\alpha_i(n, x)| < K,$$

(*c'*) $$\lim_{x\to\infty}\lim_{n\to\infty} \sum_1^n \alpha_i(n, x) = G,$$

* Frobenius's Mean, See Bromwich, "Theory of Infinite Series," p. 310.

(*d*) $\phi_i(n, x)$ *is bounded for all values of* i, n *and* x, *and the sequence* $\phi_i(n, x)$, $i = 1, 2, 3, \cdots$, *converges to* ϕ_i *uniformly in* n *and* x. *Then*

$$\lim_{x\to\infty}\lim_{n\to\infty}\sum_1^n \phi_i(n, x)\alpha_i(n, x) = C\cdot\phi.$$

For we can write

$$\lim_{x\to\infty}\lim_{n\to\infty}\sum_1^n \phi_i\alpha_i - C\cdot\phi = \lim_{x\to\infty}\lim_{n\to\infty}\sum_1^n \phi_i\alpha_i - \phi\lim_{x\to\infty}\lim_{n\to\infty}\sum_1^n \alpha_i$$
$$= \lim_{x\to\infty}\lim_{n\to\infty}\sum_1^n (\phi_i - \phi)\alpha_i,$$

and the right member of the second identity is zero, since $(\phi_i - \phi)$ satisfies assumption (*c*) of the above theorem.

If $\alpha_i(n, x)$ takes the form $f(z_i)\Delta z_i$ where $f(z)$ is absolutely integrable on the interval a to b and $\sum_1^n \Delta z_i = b - a$, and if ϕ_i is the sum to i terms of a convergent series, the conclusion of the above corollary takes the form

$$\lim_{n\to\infty}\sum_1^n f(z_i)\phi_i\Delta z_i = \lim_{i\to\infty}\phi_i\int_a^b f(z)dz.$$

This summation process is regular whenever the integral on the right is unity. Hence we can obtain any number of regular definitions of summability by choosing different functions with which it is possible to set up definite integrals equal to unity, the functions being absolutely integrable on their respective intervals. For example, from $\int_0^1 dz$ we infer the regularity of Frobenius's Mean, $\lim_{n\to\infty} 1/n\sum_1^n s_i$ and from $\int_0^{\pi/2}\sin z dz$ we infer this property of $\lim_{n\to\infty}\sum_1^n s_i(\pi/2n)\sin i\pi/2n$.

More generally we consider the following definition.

$$(S)\qquad S = \lim_{x\to\infty}\lim_{n\to\infty}\sum_1^n s_i\alpha_i(n, x),$$

where $\alpha_i(n, x)$ is restricted as follows:

$$(a')\qquad \lim_{x\to\infty}\lim_{n\to\infty}\alpha_i(n, x) = 0,$$

$$(b')\qquad \sum_1^n |\alpha_i(n, x)| \leqq K, \qquad x > X, \qquad n > N_x{}^*$$

and

$$(c'')\qquad \lim_{x\to\infty}\lim_{n\to\infty}\sum_1^n \alpha_i(n, x) = 1.$$

* That is, $\Sigma_1^n |\alpha_i|$ is bounded for x greater than a properly chosen X and n greater than N dependent upon x. This assumption is all that is essential in (*b*) of Theorem I and Corollary II.

That these restrictions are *sufficient* to make (S) regular is an obvious deduction from corollary II. Toeplitz* has proved that a set of restrictions, equivalent to the above when α_i is a function of n only are necessary and sufficient to make the definition $S = \lim_{n\to\infty} \sum_1^n s_i\alpha_i(n)$ regular. His proof of the *necessity* of these restrictions is applicable to the case we are considering, that is, when α_i is a function of both n and x. In this connection it is interesting to consider the necessity of the restrictions of (S) in order to sum a single convergent series to its ordinary sum. We omit the trivial case in which all the terms of the series are zero. It is evident that if (c'') holds, either (a') or (b') or both may fail and the series whose sum to i terms is constant will still sum to that constant. *However if (a') and (b') hold, and (S) sums, to its ordinary sum, a single convergent series whose sum is not zero, then*

$$\lim_{x\to\infty}\lim_{n\to\infty}\sum_1^n \alpha_i = 1,$$

if it exists. This is an immediate consequence of Corollary II.

An interesting special case of (S) is obtained by assuming that α_i is positive. This assumption and (c'') imply (b'). We then have the definition,

$$(S) \qquad S = \lim_{x\to\infty}\lim_{n\to\infty}\sum_1^n s_i\alpha_i(n, x),$$

where α_i is positive and satisfies (a') and (c'') of (S).

Definition (S) may sum properly divergent series† but (S') can not.

Consider the definition for which the divergence factors are

$$3/n,\ -1/n,\ 3/n,\ -1/n,\ \cdots,\ [1-2(-1)^i]/n,\ \cdots.$$

These forms for α_i satisfy the restrictions of (S), and this definition sums, to zero, the properly divergent series whose sequence of sums is

$$\sqrt{1},\ 3\sqrt{1},\ \sqrt{2},\ 3\sqrt{2},\ \cdots,\ [2+(-1)^i]\sqrt{\frac{2i+1-(-1)^i}{4}},\ \cdots,$$

for

$$\sum_1^n s_i\alpha_i = \begin{cases} 0, & n \text{ even} \\ (3/n)\sqrt{(n+1)/2}, & n \text{ odd} \end{cases}$$

The fact that (S') can not sum properly divergent series is an instance of the following general theorem.

* Toeplitz, Prace Matematyczno-fizyczno, Vol. 22 (1911), p. 113.

† A properly divergent series is one for which $\lim_{n\to\infty} s_n = \lim_{n\to\infty}\sum u_i = +\infty$ or $-\infty$. In theorem II, which follows, it is only necessary to consider the case in which $\lim_{n\to\infty} s_n = +\infty$.

Let α_i be not only a function of i, a positive integer, but of any other variables whatever and let $\lim'\text{s} \sum_1^n s_i\alpha_i$ denote the result of applying any limiting processes whatever to $\sum_1^n s_i\alpha_i$. Then we have

THEOREM II. *If the definition,*

$$S = \lim'\text{s} \sum_1^n s_i\alpha_i,$$

where $\alpha_1 \geqq 0$, *is regular, it can not sum properly divergent series.*

PROOF. Let the series whose sum to i terms is s_i properly diverge. Then for an arbitrary C, we can choose j such that $S_i \geqq C$ for every $i \geqq j$. Now consider the sequences,

$$(1) \qquad s_1,\ s_2,\ s_3,\ \cdots,\ s_j,\ s_{j+1},\ s_{j+2},\ \cdots,$$

$$(2) \qquad s_1,\ s_2,\ s_3,\ \cdots,\ s_{j-1},\ C,\ C,\ C,\ \cdots,$$

$$(3) \qquad 0,\ 0,\ 0,\ \cdots,\ s_j - C,\ s_{j+1} - C,\ s_{j+2} - C,\ \cdots.$$

The second sequence arises from a series that is summable to C. Consequently if the series from which the first series arises, that is the given series, is summable, we have from elementary limit theory

$$\lim'\text{s} \sum_1^n s_i\alpha_i - C = \lim'\text{s} \sum_j^n (s_i - C)\alpha_i.$$

Since the right member is positive however large we choose C, we conclude that $\lim'\text{s} \sum_1^n s_i\alpha_i$ does not exist.

As a direct application of Theorem I, we have the following theorem, which is not without interest for its own sake.

THEOREM III. *If the series whose sum to i terms is* s_i' *converges or oscillates between finite limits and is summable to* s', *by* (S), *and if the series whose sum to i terms is* s_i *converges to* s, *then the series whose sum to i terms is* $s_i's_i$ *sums to* ss', *by* (S).

PROOF. Since s_i' is bounded, assumptions (a) and (b) of Theorem I are satisfied by $s_i'\alpha_i$ whenever they are satisfied by α_i. Hence, in that theorem, we may replace α_i by $s_i'\alpha_i$ and ϕ_i by $(s_i - s)$ and obtain

$$\lim_{x\to\infty}\lim_{n\to\infty}\sum_1^n (s_i - s)s_i'\alpha_i = 0$$

Whence

$$\lim_{r\to\infty}\lim_{n\to\infty}\sum_1^n s_i s_i'\alpha_i = \lim_{r\to\infty}\lim_{n\to\infty}\sum_1^n s s_i'\alpha_i$$
$$= s\lim_{r\to\infty}\lim_{n\to\infty}\sum_1^n s_i'\alpha_i$$
$$= ss_i'.$$

Since many special definitions are of the form

$$(a) \qquad S = \lim_{r\to\infty}\lim_{n\to\infty}\sum_1^n u_i\delta_i(n, x),$$

where u_i is the ith term of the series to be summed, it is desirable to transform Theorem I into a form that is directly applicable to such definitions.

To accomplish this, substitute,* in Theorem I,

$$(1) \qquad \phi_i = \sum_1^i \theta_h$$

and

$$(2) \qquad \alpha_i = \begin{matrix} \delta_i - \delta_{i+1}, & i < n \\ \delta_n, & i = n \end{matrix}$$

and collect the coefficients of $\theta_1, \theta_2, \cdots$, in the conclusion. This gives

THEOREM I. *Given:*

$$(e) \qquad \lim_{r\to\infty}\lim_{n\to\infty}(\delta_i - \delta_{i+1}) = 0,$$

i fixed,

$$(f) \qquad \left[\sum_1^{n-1}|\delta_i - \delta_{i+1}| + |\delta_n|\right] < K,$$

or

$$\sum_1^{n-1}|\delta_i - \delta_{i+1}| < K'$$

and $|\delta_n| < K''$, *and*

$$(g) \qquad \left|\sum_1^n \theta_i\right| \leqq e_i,$$

where e_i, $i = 1, 2, 3, \cdots$, *is a sequence of constants with limit zero, it follows from these hypotheses that*

$$\lim_{r\to\infty}\lim_{n\to\infty}\sum_1^n \theta_i(n, x)\delta_i(n, x) = 0.$$

From (2) we see that $\delta_1 = \sum_1^n \alpha_i$. Hence (c') of Corollary II gives us

$$\lim_{r\to\infty}\lim_{n\to\infty}\delta_1 = C$$

* This substitution is suggested by collecting the coefficients $u_1, u_2, u_3, \ldots$, in $S_n = \Sigma_1^n (\Sigma_1^i u_h)\alpha_i$. The coefficient of u_i is $\Sigma_i^n \alpha_h$ $(= \delta_i)$.

which taken with (e) and (f) give (e') and (f') of the following transform of Corollary II.

COROLLARY II′. *Given:*

$$(e') \qquad \lim_{x\to\infty} \lim_{n\to\infty} \delta_i = C,$$

i fixed.

$$(f') \qquad \sum_1^{n-1} |\delta_i - \delta_{i+1}| < K, \qquad x > X, \qquad n > N_x,$$

and

$$(g') \qquad \sum_1^i \theta_h(n, x)$$

is bounded for all values of i, n *and* x, *and*

$$\sum_1^i \theta_h(n, x)$$

converges to θ *uniformly in* n *and* x, *then*

$$\lim_{x\to\infty} \lim_{n\to\infty} \sum_1^n \theta_i \delta_i = C \cdot \theta.$$

Consider now the definition

$$(u) \qquad S = \lim_{x\to\infty} \lim_{n\to\infty} \sum_1^n u_i \delta_i,$$

where

$$(e'') \qquad \lim_{x\to\infty} \lim_{n\to\infty} \delta_i = 1,$$

$$(f'') \qquad |\delta_i - \delta_{i+1}| < K,$$

$x > X$ and $n > N_x$, and the definition, (u'), obtained by replacing (f'') by the assumption

$$(f''')\dagger \qquad \delta_1, \delta_2, \delta_3, \cdots,$$

is a positive, decreasing sequence. These are the definitions into which (S) and (S'), respectively, transform. Hence they possess all the properties that the latter possess. For instance, the restrictions of (u) are *necessary and sufficient* to make it regular. A similar transformation of Theorem II is easily made and is not without interest.

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* Similarly we could have used $\lim_{x\to\infty} \lim_{n\to\infty} \Sigma_j^n \alpha_i = C$, i fixed, instead of (a) and (c') of Corollar II, p. 4.

† L. L. Smail discusses this definition in these Annals, Vol. 20 (1918).

ON THE CONSISTENCY AND EQUIVALENCE OF CERTAIN GENERALIZED DEFINITIONS OF THE LIMIT OF A FUNCTION OF A CONTINUOUS VARIABLE.

By L. L. Silverman.

§ 1. **Introduction.*** In a recent paper Hurwitz and Silverman† have studied certain definitions of summability of sequences, and have obtained criteria for the consistency and for the equivalence of these definitions. Two definitions are *consistent* if whenever each of the definitions gives a value to a sequence, the two values are the same; two consistent definitions are *equivalent* when each evaluates every sequence evaluated by the other. In the present paper‡ corresponding criteria are obtained for the case of certain generalized definitions of a limit of a function of a continuous variable.

In a previous article§ the author has studied the transformation||

$$(1) \qquad v(x) = \alpha u(x) + \int_0^x k(x, y)u(y)dy,$$

where $u(x)$ is bounded and integrable, $0 \leq x \leq x_1$, $k(x, y)$ is integrable in y for each x, $0 < \delta \leqq y \leqq x$, and $\int_0^x |k(x, y)|dy$ converges¶ for each x. The function $k(x, y)$ is the *kernel* and the number α the *coefficient* of the transformation. We shall denote by the symbol $[\alpha, k(x, y)]$ the transformation whose coefficient is α and whose kernel is $k(x, y)$. The transformation is regular if the existence of

$$\lim_{x=\infty} u(x)$$

implies the existence of

$$\lim_{x=\infty} v(x)$$

and the equality of the limits. Examples of regular transformations are the identity

$$(E) \qquad \alpha = 1, \qquad k(x, y) = 0$$

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† Transactions of the American Mathematical Society, vol. 18 (1917), p. 1.

‡ The author is greatly indebted to Professor W. A. Hurwitz for many valuable suggestions in connection with this paper.

§ Transactions of the American Mathematical Society, vol. 17 (1916), p. 284.

|| The lower limit of integration might be any number, but is taken zero for convenience.

¶ The convergence of the integral is assumed in order to give a meaning to formula 1.

and the transformation

$$(M) \qquad \alpha = 0, \qquad k(x, y) = \frac{1}{x}, \qquad 0 < x.$$

The transformations E and M correspond in the case of sequences to convergence and to summability of the first order, respectively. To obtain the analog of Cesàro-summability of higher orders, we define with Landau*

$$v_0(x) = u(x), \qquad v_n(x) = \frac{n!}{x^n} S_n(x), \qquad x > 0,$$

where

$$S_0(x) = u(x), \qquad S_n(x) = \int_0^x S_{n-h}(y)dy; \qquad n = 1, 2, 3, \cdots.$$

Given $u(x)$, we can by this definition find $v_n(x)$. To obtain the functional relation between $v_n(x)$ and $u(x)$ in the form (1), we shall prove the formula

$$(2) \qquad v_n(x) = \frac{n!}{x^n}\int_0^x \frac{(x-y)^{h-1}}{(h-1)!} S_{n-h}(y)dy, \qquad h = 1, 2, 3, \cdots, n.$$

Let n be any fixed integer, and assume that formula (2) holds for some fixed h; then expressing S_{n-h} in terms of S_{n-h-1}, we have

$$\begin{aligned} v_n(x) &= \frac{n!}{x^n}\int_0^x \int_0^y \frac{(x-y)^{h-1}}{(h-1)!} S_{n-h-1}(z)dzdy \\ &= \frac{n!}{x^n}\int_0^x \int_z^x \frac{(x-y)^{h-1}}{(h-1)!} S_{n-h-1}(z)dydz \\ &= \frac{n!}{x^n}\int_0^x \frac{(x-z)^{h}}{h!} S_{n-h-1}(z)dz, \end{aligned}$$

which is formula (2) when h is replaced by $h + 1$. Since the formula obviously holds for $h = 1$, it is true in general. Letting $h = n$ in (2), we obtain

$$v_n(x) = \frac{n!}{x^n}\int_0^x \frac{(x-y)^{n-1}}{(n-1)!} u(y)dy,$$

which is of the form (1). We have thus found

$$(C_n) \qquad \alpha = 0, \qquad c_n(x, y) = \frac{n(x-y)^{n-1}}{x^n}, \qquad 0 < x, \qquad 0 \leqq y \leqq x,$$

the coefficient and the kernel of the transformation C_n, corresponding to the case of Cesàro summability of order n.

* Sächsische Berichte, vol. 65 (1913), p. 131.

The analog to Hölder summability is defined* by the formulæ

$$v_0(x) = u(x), \qquad v_n(x) = \frac{1}{x}\int_0^x v_{n-1}(y)dy, \qquad n = 1, 2, 3, \cdots.$$

Since $u(x)$ is bounded, $0 \leqq x \leqq x_1$, it follows that $v_n(x)$ is bounded and continuous, $0 \leqq x \leqq x_1$, $n = 1, 2, \cdots$. To obtain the functional relation between $v_n(x)$ and $u(x)$ in the form (1), we shall prove the formula

$$(3) \quad v_n(x) = \frac{1}{(h-1)!\,x}\int_0^x \log^{h-1}\left(\frac{x}{y}\right) v_{n-h}(y)dy, \qquad h = 1, 2, 3, \cdots, n.$$

We shall first show that the integral in (3) converges. If M is an upper bound of $u(x)$, we have

$$\frac{1}{(h-1)!\,x}\int_0^x \log^{h-1}\left(\frac{x}{y}\right)|v_{n-h}(y)|dy \leqq \lim_{\delta=0}\frac{M}{(h-1)!\,x}\int_\delta^x \log^{h-1}\left(\frac{x}{y}\right)dy$$

$$= \lim_{\delta=0}\left[M\frac{y}{x}\sum_{p=0}^{h-1}\frac{\log^p\left(\frac{x}{y}\right)}{p!}\right]_{y=\delta}^{y=x} = M[1-0] = M,$$

so that the integral in (3) converges. Now let n be any fixed number, and assume that formula (3) holds for some h; then expressing v_{n-h} in terms of v_{n-h-1}, we have

$$v_n(x) = \lim_{\delta=0}\frac{1}{(h-1)!\,x}\int_\delta^x\int_0^y \frac{1}{y}\log^{h-1}\left(\frac{x}{y}\right)v_{n-h-1}(z)dzdy$$

$$= \lim_{\delta=0}\frac{1}{(h-1)!\,x}\int_\delta^x\int_z^x \frac{1}{y}\log^{h-1}\left(\frac{x}{y}\right)v_{n-h-1}(z)dydz$$

$$+ \lim_{\delta=0}\frac{1}{(h-1)!\,x}\int_\delta^x\int_0^\delta \frac{1}{y}\log^{h-1}\left(\frac{x}{y}\right)v_{n-h-1}(z)dzdy.$$

Since

$$\lim_{\delta=0}\left|\frac{1}{(h-1)!\,x}\int_\delta^x\int_0^\delta \frac{1}{y}\log^{h-1}\left(\frac{x}{y}\right)v_{n-h-1}(z)dzdy\right|$$

$$\leqq \lim_{\delta=0}\frac{M}{(h-1)!\,x}\int_0^\delta\int_\delta^x \frac{1}{y}\log^{h-1}\left(\frac{x}{y}\right)dydz = \lim_{\delta=0}\frac{M\delta}{h!\,x}\log^h\left(\frac{x}{\delta}\right) = 0,$$

it follows that

$$v_n(x) = \frac{1}{(h-1)!\,x}\int_0^x v_{n-h-1}(z)\int_z^x \frac{1}{y}\log^{h-1}\left(\frac{x}{y}\right)dydz$$

$$= \frac{1}{h!\,x}\int_0^x \log^h\left(\frac{x}{z}\right)v_{n-h-1}(z)dz,$$

* Landau, loc. cit.

which is formula (3) when h is replaced by $h+1$. Since the formula holds for $h = 1$, it is true in general. Letting $h = n$ in (3), we obtain

$$v_n(x) = \frac{1}{(n-1)!\,x}\int_0^x \log^{n-1}\left(\frac{x}{y}\right)u(y)dy, \qquad 0 < y \leqq x,$$

which is of the form (1). We have thus found

$$(H_n) \quad \alpha = 0, \qquad h_n(x, y) = \frac{1}{(n-1)!\,x}\log^{n-1}\left(\frac{x}{y}\right), \qquad 0 < y \leqq x,$$

the coefficient and the kernel of the transformation H_n, corresponding to the case of Hölder summability of order n.

Let $k(x, y)$ be integrable in y for each x, $0 < y \leqq x$; then *a sufficient condition that* $[\alpha, k(x, y)]$ *correspond to a regular transformation* is*

$$(a) \qquad |k(x, y)| \leqq \frac{N}{x^{1-\rho}y^{\rho}},$$

$$(b) \qquad \lim_{x=\infty}\int_0^x k(x, y)dy = 1 - \alpha,$$

where $0 \leqq \rho < 1$ *and* $N \geqq 0$.

It is easily verified that the transformations C_n and H_n satisfy these conditions for regularity. We shall be concerned in this paper with only those kernels which satisfy conditions (a) and (b).

When the integral equation (1) possesses a solution in the form

$$(2) \qquad u(x) = \beta v(x) + \int_0^x l(x, y)v(y)dy,$$

we shall call $l(x, y)$ the *inverse kernel*,† and the transformation (2) the inverse *transformation*; symbolically, if $v(x) = A[u(x)]$, then

$$u(x) = A^{-1}[v(x)].$$

If $v(x) = A[u(x)]$ and $w(x) = B[v(x)]$, then $w(x) = B[A(u(x)]$, the transformation being BA. If A and B correspond to $[\alpha, k(x, y)]$ and $[\beta, l(x, y)]$ respectively, $aA + bB$ will correspond to $[a\alpha + b\beta, ak(x, y) + bl(x, y)]$. If $A_1, A_2, \cdots$ correspond to $[\alpha_1, k_1(x, y)]$, $[\alpha_2, k_2(x, y)]$, $\cdots$, respectively, $a_1A_1 + a_2A_2 + \cdots$ will correspond to $[\alpha, k(x, y)]$, whenever

$$\lim_{n=\infty}[a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n] = \alpha,$$

$$\lim_{n=\infty}[a_1k_1(x, y) + a_2k_2(x, y) + \cdots + a_nk_n(x, y)] = k(x, y),$$

the last limit existing uniformly‡ in y for each x.

* Bulletin of the American Mathematical Society, vol. 22, p. 461. For other forms of necessary and sufficient conditions see article in Transactions, vol. 17.

† In the theory of integral equations the kernel in (1) is usually defined to be $-(1/\alpha)\,k(x, y)$, and the inverse kernel, usually called the resolvent function, is then $\alpha l(x, y)$.

‡ The uniform approach of this limit is assumed in order to ensure the integrability of $k(x, y)$.

§ 2. **A special class of transformations.** It will be seen that if A and B are regular, then for any constant a, AB and $aA + (1 - a)B$ are regular. Landau* has studied the regular transformation

$$aE + (1 - a)M.$$

It is natural to consider the more general regular transformation

$$a_0E + a_1M + a_2M^2 + \cdots + a_nM^n, \qquad a_0 + a_1 + a_2 + \cdots + a_n = 1,$$

or, more generally, the symbol

$$a_0E + a_1M + a_2M^2 + \cdots + a_nM^n + \cdots,$$

and to ask under what conditions the symbol defines a regular transformation.

THEOREM 1. *If $f(z) = a_0 + a_1z + a_2z^2 + \cdots$ is analytic within and on the boundary of the unit circle, and if $f(1) = 1$, then the symbol $a_0E + a_1M + a_2M^2 + \cdots$ defines a regular transformation.*

To prove this theorem we shall show that the sufficient conditions for regularity (a) and (b) are satisfied. Starting with the inequality†

$$\frac{1}{n!}\log^n(t) \leqq \frac{1}{\rho^n}t^\rho, \qquad t \geqq 1, \qquad 0 < \rho < 1, \qquad n = 1, 2, 3, \cdots,$$

letting $t = x/y$, and dividing by x, we have

$$h_{n+1}(x, y) = \frac{1}{n!\,x}\log^n\left(\frac{x}{y}\right) \leqq \frac{1}{\rho^n}\frac{1}{x^{1-\rho}y^\rho}, \qquad 0 < y \leqq x, \qquad 0 < \rho < 1;$$

so that

$$|k(x, y)| \leqq \Sigma\,|a_n|\,h_n(x, y) \leqq \frac{\rho}{x^{1-\rho}y^\rho}\sum_{n=1}^{\infty}\frac{|a_n|}{\rho^n},$$

where the series

$$\sum_{n=1}^{\infty}\frac{|a_n|}{\rho^n}$$

surely converges for values of ρ sufficiently near unity, since $\sum_{n=0}^{\infty} a_nz^n$ is analytic on the circumference of the unit circle. Letting the value of the series $\sum_{n=1}^{\infty}|a_n|/\rho^n$, for a fixed admissible ρ, equal to m, we have

$$|k(x, y)| \leqq \frac{m\rho}{x^{1-\rho}y^\rho}$$

which shows that condition (a) is satisfied.

* Sächsische Berichte, loc. cit.

† This inequality is easily proved by mathematical induction. Assuming it to hold for $n = k$, dividing by t, and integrating from 1 to t, we see that it holds for $n = k + 1$. To show that it holds for $n = 1$, we observe that for $t = 1$, we get $0 \leqq 1/\rho$, and for $t > 1$, the derivative of the left is less than the derivative of the right side of the inequality.

To prove that condition (*b*) is satisfied, we observe that, since

$$h_{n+1}(x, y) \leqq \frac{1}{\rho^n}\frac{1}{x^{1-\rho}y^\rho}, \qquad 0 < \delta \leqq y \leqq x, \qquad 0 < \rho < 1,$$
$$\leqq \frac{1}{\rho^n}\frac{1}{x^{1-\rho}\delta^\rho}$$

the series

$$\sum_{n=1}^\infty a_n h_n(x, y) = k(x, y)$$

converges uniformly for each x, $\delta \leqq y \leqq x$. Hence we may integrate term by term, and

$$\int_\delta^x k(x, y)dy = \sum_{i=1}^\infty a_i \int_\delta^x h_i(x, y)dy,$$

and the series of integrals converges uniformly in δ. We accordingly have*

$$\lim_{\delta \doteq 0} \int_\delta^x k(x, y)dy = \sum_{i=1}^\infty a_i \lim_{\delta \doteq 0} \int_\delta^x h_i(x, y)dy$$
$$= \sum_{i=1}^\infty a_i = f(1) - a_0 = 1 - a_0.$$

Corollary 1. *The kernel of the transformation* M^n *is given by the formula*

$$(4) \qquad h_n(x, y) = \frac{1}{2\pi i x}\int_C t^{n-2}\left(\frac{x}{y}\right)^{1/t} dt, \qquad 0 < x, \qquad 0 < y \leqq x,$$

where C *is any circle including the origin.*

For

$$\frac{1}{2\pi i x}\int_C t^{n-2}\left(\frac{x}{y}\right)^{1/t} dt = \frac{1}{2\pi i x}\int_C t^{n-2}\sum_{p=0}^\infty \frac{1}{p!}\frac{1}{t^p}\log^p\left(\frac{x}{y}\right) dt$$
$$= \frac{1}{2\pi i x}\sum_{p=0}^\infty \frac{1}{p!}\log^p\left(\frac{x}{y}\right)\int_C t^{n-2-p}dt,$$

since the first series cnoverges uniformly in t, its general term being less in absolute value than $N^p \log^p (x/y)/p!$, where N is the maximum of $|1/t|$ on C. Since

$$\int_C t^{n-2-p}dt = 0, \qquad p \neq n - 1,$$
$$= 2\pi i, \qquad p = n - 1,$$

* By a well-known theorem (see Osgood, Funktionentheorie, vol. 1, 2d ed., p. 593), the order of two limits, to be performed in succession, may be interchanged, provided each limit exists and one of them exists uniformly.

it follows that

$$\frac{1}{2\pi i x}\int_C t^{n-2}\left(\frac{x}{y}\right)^{1/t} dt = \frac{1}{x}\frac{1}{(n-1)!}\log^{n-1}\left(\frac{x}{y}\right) = h_n(x, y).$$

COROLLARY 2. *The transformation* $[\alpha, k(x, y)]$ *is given in terms of* $f(z)$, *a function analytic in a circle* C *including the origin, by the formulae,*

$$(5) \quad \alpha = f(0), \quad k(x, y) = \frac{1}{2\pi i x}\int_{C_1}\frac{f(t)}{t^2}\left(\frac{x}{y}\right)^{1/t} dt, \quad 0 < x, \quad 0 < y \leqq x,$$

where C_1 *is a circle including the origin entirely inside* C.

Obviously $\alpha = a_0 = f(0)$; it remains to derive the expression for $k(x, y)$. If $f(z) = \sum_{n=0}^{\infty} a_n z^n$, then by the preceding corollary*

$$\begin{aligned} k(x, y) &= \sum_{n=1}^{\infty} a_n h_n(x, y) = \frac{1}{2\pi i x}\sum_{n=0}^{\infty}\int_{C_1} a_n t^{n-2}\left(\frac{x}{y}\right)^{1/t} dt \\ &= \frac{1}{2\pi i x}\int_{C_1}\frac{1}{t^2}\sum_{n=1}^{\infty} a_n t^n\left(\frac{x}{y}\right)^{1/t} dt \\ &= \frac{1}{2\pi i x}\int_{C_1}\frac{1}{t^2}[f(t) - a_0]\left(\frac{x}{y}\right)^{1/t} dt \\ &= \frac{1}{2\pi i x}\int_{C_1}\frac{f(t)}{t^2}\left(\frac{x}{y}\right)^{1/t} dt. \end{aligned}$$

Definition. The transformation A *corresponds* to $f(z)$, a function analytic in some circle C including the origin, if the coefficient α and the kernel $k(x, y)$ of the transformation are given by (5).

COROLLARY 3. *The kernel of the transformation corresponding to*

$$g(z) = \frac{n!\, z^n}{(1+z)(1+2z)\cdots(1+\overline{n-1}\, z)}$$

is

$$C_n(x, y) = \frac{n}{x}\left(1 - \frac{y}{x}\right)^{n-1}, \quad 0 < x, \quad 0 \leqq y \leqq x.$$

For let $k(x, y)$ be the kernel corresponding to $g(z)$; then by the preceding corollary

$$k(x, y) = \frac{n!}{2\pi i x}\int_{C_1}\frac{t^{n-2}}{(1+t)(1+2t)\cdots(1+\overline{n-1}\, t)}\left(\frac{x}{y}\right)^{1/t} dt$$

provided C_1 is a circle about the origin of radius less than $1/(n-1)$.

* Term by term integration is justified by the uniform convergence of the series whose general term $a_n t^n (x/y)^{1/t}$ is less in absolute value than $|a_n| b^n (x/y)^c$, where b, the maximum of $|t|$ on C_1, lies inside C, and $c \geqq R(1/t)$ on C_1.

Since*

$$\frac{t^{n-2}}{(1+t)(1+2t)\cdots(1+\overline{n-1}\,t)}$$

$$= \sum_{j=1}^{n-1}(-1)^{j-1}\frac{1}{(j-1)!(n-j-1)!}\frac{1}{(jt+1)},$$

hence

$$k(x,y) = \frac{n!}{2\pi i x}\int_{C_1}\sum_{j=1}^{n-1}(-1)^{j-1}\frac{1}{(j-1)!(n-j-1)!}\frac{1}{jt+1}\left(\frac{x}{y}\right)^{1/t}dt$$

$$= \frac{n!}{2\pi i x}\sum_{j=1}^{n-1}(-1)^{j-1}\frac{1}{(j-1)!(n-j-1)!}\int_{C_1}\frac{1}{jt+1}\left(\frac{x}{y}\right)^{1/t}dt.$$

Letting $t = 1/s$ and C' the inverse of C_1, we have

$$\int_{C_1}\frac{1}{jt+1}\left(\frac{x}{y}\right)^{1/t}dt = \int_{C'}\frac{1}{(j+s)s}\left(\frac{x}{y}\right)^{s}ds = \frac{2\pi i}{j}\left(1-\frac{y^j}{x^j}\right).$$

Hence

$$k(x,y) = \frac{n!}{2\pi i x}\sum_{j=1}^{n-1}(-1)^{j-1}\frac{1}{(j-1)!(n-j-1)!}\frac{2\pi i}{j}\left(1-\frac{y^j}{x^j}\right)$$

$$= \frac{n}{x}\sum_{j=1}^{n-1}(-1)^{j-1}\frac{(n-1)!}{j!(n-j-1)!}\left[1-\left(\frac{y}{x}\right)^j\right]$$

$$= \frac{n}{x}\sum_{j=0}^{n-1}(-1)^{j}\frac{(n-1)!}{j!(n-j-1)!}\left(\frac{y}{x}\right)^j$$

$$= \frac{n}{x}\left(1-\frac{y}{x}\right)^{n-1} = C_n(x,y).$$

THEOREM 2. *If A and B correspond to $f(z)$ and $g(z)$ respectively, then BA corresponds to $g(z)f(z)$.*

Let C correspond to $g(z)f(z)$; then we are to prove that $C = BA$. If $k(x, y)$, $l(x, y)$, $m(x, y)$ are the kernels of A, B, C respectively, then the transformations A and BA are given as follows:

$$v(x) = f(0)u(x) + \int_0^x k(x,y)u(y)dy,$$

$$w(x) = g(0)v(x) + \int_0^x l(x,y)v(y)dy$$

$$= g(0)f(0)u(x) + \int_0^x [g(0)k(x,y) + f(0)l(x,y)]u(y)dy$$

$$+ \int_0^x\int_0^y l(x,y)k(y,s)u(s)dsdy.$$

* This formula is easily obtained by resolving into partial fractions.

Interchanging the order of integration* in the double integral, and letting

$$q(x, s) = \int_s^x l(x, y)k(y, s)dy, \tag{6}$$

we obtain

$$w(x) = g(0)f(0)u(x) + \int_0^x [g(0)k(x, y) + f(0)l(x, y) + q(x, y)]u(y)dy.$$

Since the coefficient of the transformation BA is seen to be $g(0)f(0)$, which is the same as that of the transformation C, it remains only to prove the kernels of C and BA are identical, i.e.,

$$m(x, y) = g(0)k(x, y) + f(0)l(x, y) + q(x, y). \tag{7}$$

From (5) and (6) we obtain

$$\begin{aligned} -4\pi^2 xq(x, y) &= -4\pi^2 x \int_y^x l(x, s)k(s, y)ds \\ &= \int_y^x \left[\int_{C_1} \frac{g(t_1)}{t_1^2} \left(\frac{x}{s} \right)^{1/t_1} dt_1 \right] \frac{1}{s} \int_{C_2} \frac{f(t_2)}{t_2^2} \left(\frac{s}{y} \right)^{1/t_2} dt_2 ds \\ &= \int_{C_1} \frac{g(t_1)}{t_1^2} x^{1/t_1} \int_{C_2} \frac{f(t_2)}{t_2^2} y^{-1/t_2} \int_y^x s^{1/t_2 - 1/t_1 - 1} ds dt_2 dt_1 \\ &= \int_{C_1} \int_{C_2} \frac{g(t_1)f(t_2)}{t_1 t_2(t_1 - t_2)} \left[\left(\frac{x}{y} \right)^{1/t_2} - \left(\frac{x}{y} \right)^{1/t_1} \right] dt_2 dt_1, \end{aligned}$$

where C_1 and C_2 are circles including the origin. Taking C_1 inside C_2, we have

$$\begin{aligned} \int_{C_1} \int_{C_2} \frac{g(t_1)f(t_2)}{t_1 t_2(t_1 - t_2)} \left(\frac{x}{y} \right)^{1/t_2} dt_2 dt_1 &= \int_{C_2} \frac{f(t_2)}{t_2} \left(\frac{x}{y} \right)^{1/t_2} \left[\int_{C_1} \frac{g(t_1)}{t_1(t_1 - t_2)} dt_1 \right] dt_2 \\ &= -2\pi i g(0) \int_{C_2} \frac{f(t_2)}{t_2^2} \left(\frac{x}{y} \right)^{1/t_2} dt_2 \end{aligned}$$

and

$$\begin{aligned} \int_{C_1} \int_{C_2} \frac{g(t_1)f(t_2)}{t_1 t_2(t_1 - t_2)} \left(\frac{x}{y} \right)^{1/t_1} &= -\int_{C_1} \frac{g(t_1)}{t_1} \left(\frac{x}{y} \right)^{1/t_1} \left[\int_{C_2} \frac{f(t_2)}{t_2(t_2 - t_1)} dt_2 \right] dt_1 \\ &= -2\pi i \int_{C_1} \frac{g(t_1)}{t_1} \left(\frac{x}{y} \right)^{1/t_1} \left[\frac{f(t_1) - f(0)}{t_1} \right] dt_1. \end{aligned}$$

* The change in the order of integration is easily justified if we bear in mind that the kernels $k(x, y)$, $l(x, y)$ are subject to the conditions of p. 131.

Hence

$$- 4\pi^2 xq(x, y) = - 2\pi i \int_C \frac{g(0)f(t) + f(0)g(t)}{t^2} \left(\frac{x}{y}\right)^{1/t} dt$$

$$+ 2\pi i \int_C \frac{g(t)f(t)}{t^2} \left(\frac{x}{y}\right)^{1/t} dt,$$

or

$$- 4\pi^2 xq(x, y) = 4\pi^2 x[g(o)k(x, y) + f(0)l(x, y)] - 4\pi^2 xm(x, y)$$

which reduces to (7).

Corollary. *If A and B correspond to $f(z)$ and $g(z)$ respectively, each of the functions being analytic in some circle including the origin, then $AB = BA$.*

§ 3. **A more general class of transformations.** We have seen in the preceding section that if the function $f(z)$ is analytic in the unit circle, then the corresponding transformation is regular. We shall now consider more generally the function $f(z)$ analytic in a circle C, of radius $\frac{1}{2}$ about the point $\frac{1}{2}$, and define the corresponding transformation $f(M)$ by (5), where the circle C_1 is now to be taken of radius $\frac{1}{2} + \epsilon$ ($\epsilon > 0$) about the point $\frac{1}{2}$.

Theorem 3. *If $f(z)$ is analytic inside and on the boundary of the circle of radius $\frac{1}{2}$ about the point $\frac{1}{2}$, and if $f(1) = 1$, then the transformation corresponding to $f(z)$ is regular.*

We shall show that the sufficient conditions for the regularity of a transformation, referred to in the first section, are satisfied. By (5) we have

$$\int_\delta^x k(x, y)dy = \frac{1}{2\pi i x}\int_\delta^x \int_{C_1} \frac{f(t)}{t^2}\left(\frac{x}{y}\right)^{1/t} dtdy$$

$$= \frac{1}{2\pi i x}\int_{C_1} \frac{f(t)}{t^2}\int_\delta^x \left(\frac{x}{y}\right)^{1/t} dydt$$

$$= \frac{1}{2\pi i}\int_{C_1} \frac{f(t)}{t^2\left(1 - \frac{1}{t}\right)}\left[1 - \left(\frac{\delta}{x}\right)^{1-1/t}\right] dt.$$

Now $|f(t)/t^2(1 - 1/t)| = |f(t)/t(t - 1)|$ is continuous along C_1 and accordingly has a maximum N. Hence

$$\left|\int_{C_1} \frac{f(t)}{t(t - 1)}\left(\frac{\delta}{x}\right)^{1-1/t} dt\right| \leqq N\int_{C_1}\left|\left(\frac{\delta}{x}\right)^{1-1/t}\right| |dt|$$

$$= N\int_{C_1}\left(\frac{\delta}{x}\right)^{R(1-1/t)} |dt| \leqq 2\pi N\left(\frac{\delta}{x}\right)^{1+1/\epsilon} (\tfrac{1}{2} + \epsilon);$$

for, since t lies on the circumference of C_1,

$$\frac{\epsilon}{1+\epsilon} \leqq R\left(1 - \frac{1}{t}\right) \leqq 1 + \frac{1}{\epsilon}.$$

We have also

$$\int_{C_1} \frac{f(t)}{t(t-1)}\, dt = 2\pi i[f(1) - f(0)].$$

Hence

$$\int_0^x k(x, y)dy = \lim_{\delta=0} \int_\delta^x k(x, y)dy$$

$$= f(1) - f(0) = 1 - f(0) = 1 - \alpha,$$

since by definition $\alpha = f(0)$. Thus condition (b) is satisfied.

Secondly, letting N_1 represent the maximum of $f(t)/t^2$ on C_1, we have

$$|k(x, y)| \leqq \frac{1}{2\pi x} \int_{C_1} \left|\frac{f(t)}{t^2}\right| \left|\frac{x}{y}\right|^{1/t} |dt| \leqq \frac{N_1}{2\pi x} \int_{C_1} \left(\frac{x}{y}\right)^{R(1/t)} |dt|.$$

Since

$$R\left(\frac{1}{t}\right) \leqq \frac{1}{1+\epsilon},$$

we obtain

$$|k(x, y)| \leqq \frac{N_1}{2\pi x} \int_{C_1} \left(\frac{x}{y}\right)^{1/1+\epsilon} |dt|$$

$$= \frac{N_1(\frac{1}{2} + \epsilon)}{x^{\epsilon/1+\epsilon} y^{1/1+\epsilon}}.$$

Hence, letting

$$N_1(\tfrac{1}{2} + \epsilon) = N, \qquad \frac{1}{1+\epsilon} = \rho,$$

$$|k(x, y)| \leqq \frac{N}{x^{1-\rho} y^\rho}, \qquad 0 < \rho < 1,$$

which is condition (a). The transformation has accordingly been shown to be regular.

We shall now consider the function $f(z)$ analytic in C except for poles, and study the corresponding symbol $f(M)$.

Lemma 1. The function $f(z) = (1 - \rho)/(z - \rho)$, where ρ is a point inside or on the boundary of the circle C, does not define a regular transformation.

The proposition is obviously true for the cases $\rho = 0$ and $\rho = 1$. In every other case the function $f(z)$ defines a transformation $f(M)$, which we shall show is not regular. To show this, define

$$v(x) = x^p, \qquad p = \frac{1}{\rho} - 1,$$

and find $u(x)$ by applying to $v(x)$ the transformation corresponding to

$$\frac{1}{f(z)} = \frac{z-\rho}{1-\rho}.$$

Thus

$$u(x) = \frac{-\rho}{1-\rho}v(x) + \frac{1}{(1-\rho)x}\int_0^x v(y)dy$$

$$= \frac{-\rho}{1-\rho}x^p + \frac{1}{(1-\rho)x}\int_0^x y^p dy$$

$$= \left(-\rho + \frac{1}{p+1}\right)\frac{x^p}{1-\rho} = 0.$$

Hence

$$\lim_{x=\infty} u(x) = 0.$$

On the other hand, from the restrictions on ρ,

$$R(p) \geqq 0;$$

accordingly, since

$$\lim_{x=\infty} |x^p| = \infty, \qquad R(p) > 0$$

$$|x^p| = 1, \qquad R(p) = 0,$$

it follows that

$$\lim_{x=\infty} v(x)$$

is not zero. Hence the function $(1-\rho)/(z-\rho)$ does not define a regular transformation.

THEOREM 4. *If $f(z)$ has at least one pole in C, but is analytic except for poles within and on the boundary of C, and if $f(1) = 1$, then $f(z)$ does not define a regular transformation.*

The proof of this theorem, based on function-theoretical considerations, is identical with the one given for the case of sequences in the paper referred to above,* and is accordingly omitted here.

§ 4. **Consistency and Equivalence of Regular Transformation.** We shall now use the term *analytically regular* to describe a transformation $f(M)$ corresponding to a function $f(z)$, analytic throughout C, and such that $f(1) = 1$. A number of properties of such transformations follows immediately from the results of the preceding sections.

THEOREM 5. *All analytically regular transformations are consistent.*

Let A and B, two analytically regular transformations, evaluate $\lim_{x=\infty} u(x)$ to u_1 and u_2 respectively; then BA and AB evaluate this limit to u_1 and u_2 respectively. By the corollary to Theorem 2, $AB = BA$; hence $u_1 = u_2$.

* Transactions (1917), p. 13.

THEOREM 6. *If $f(M)$ and $g(M)$ are analytically regular transformations, a necessary and sufficient condition that $f(M)$ should evaluate every expression $\lim_{x=\infty} u(x)$ which $g(M)$ evaluates, giving it the same value, is that all the zeros of $g(z)$ in C, should be zeros of at least as high order of $f(z)$.*

The proof of this theorem and that of the next theorem are omitted, being the same as those in the case of sequences in the paper already mentioned.

THEOREM 7. *If $f(M)$ and $g(M)$ are analytically regular, a necessary and sufficient condition that they be equivalent is that $f(z)$ and $g(z)$ have in C the same zeros with the same orders.*

COROLLARY 1. *A necessary and sufficient condition that the analytically regular transformation $f(M)$ be reversible (equivalent to the identical transformation) is that $f(z)$ does not vanish in C.*

COROLLARY 2. *The Hölder and Cesàro transformations of like order are equivalent.*

From corollaries 1 and 3 of Theorem 1, the transformations H_n, C_n correspond respectively to the functions

$$z^n, \frac{n!\,z^n}{(1+z)(1+2z)\cdots(1+\overline{n-1}\,z)}.$$

Each of these functions is analytic in C, having no zero except $z = 0$, which is in both cases a zero of order n. Hence the two transformations are equivalent.

A GREEN'S THEOREM IN TERMS OF LEBESGUE INTEGRALS.

By H. E. Bray.

The present paper contains the proof of the Green's Theorem which is associated with the integral form of Poisson's equation, that is to say, with the equation:

$$\int_c \frac{\partial u}{\partial x} dy - \frac{\partial u}{\partial y} dx = \int \int_R f(x, y) dx dy.$$

The method of proof, that of approximating polynomials, was suggested by Professor G. C. Evans, who uses it in the proof of a similar theorem,* where, however, the functions u, $\partial u/\partial x$, f, etc., have to obey certain restrictions as to continuity owing to the fact that the integrals in the equation are of the ordinary kind. Here, however, Lebesgue integrals are used; consequently, as might be expected, the properties of u, $\partial u/\partial x$, f, etc., are less restricted.

The region, R, here considered is a rectangle.

The works cited in footnotes are the Cours d'analyse of de la Vallée-Poussin and the article Sur l'integrale de Lebesgue by the same author in the Transactions of the American Mathematical Society, Volume XVI, 1916. They are referred to, briefly, as Cours d'analyse and Transactions respectively.

§ 1. The following theorems and definitions will be used in the course of this discussion.

Theorem A.† If the transformation

$$x_1 = \varphi(y_1, y_2), \qquad x_2 = \psi(y_1, y_2)$$

establishes a one-to-one continuous correspondence between two measurable sets, E_x and E_y, and if, moreover, the formulæ of the transformation are differentiable on E_y, at every point of E_y; and if $f(x_1, x_2)$ is summable in E_x, then $F \cdot |J|$‡ will be summable on E_y, and

$$\int_{Ex} f(x_1, x_2) dP_x = \int_{Ey} f|J| dP_y$$

provided that we agree to put $f \cdot |J| = 0$ at every point where $|J|$ vanishes, even when f becomes infinite.

* Cambridge Colloquium Lectures, September, 1916.

† De la Vallee Poussin, Transactions, p. 500.

‡ J is the Jacobian of the transformation.

The theorem can be extended to the case of more than two variables.

THEOREM *B*. If $u(x, y)$ is an absolutely continuous function of y for every value of x $(a \leq x \leq b,\ c \leq y \leq d)$, and is summable linearly with regard to x, for every value of y, and if $\partial u/\partial y$ is summable superficially in the same region, then

$$f(y) = \int_a^b u(x, y)dx$$

is an absolutely continuous function of y.

For since $\partial u/\partial y$ is summable superficially, and u linearly,

$$f(y'') - f(y') = \int_a^b [u(x, y'') - u(x, y')]dx$$

$$= \int_a^b dx \int_{y'}^{y''} \frac{\partial u}{\partial y} dy = \int_{R(a,\, b;\, y'y'')} \int \frac{\partial u}{\partial y} dxdy.$$

Therefore

$$\sum_{i=1}^{\infty} [f(y_{i+1}) - f(y_i)] = \int_E \int \frac{\partial u}{\partial x} dxdy,$$

where E is a denumerable set of rectangles $R_i[y_i \leq y \leq y_{i+1},\ a \leq x \leq b]$. Since the double integral is absolutely continuous, $\sum_{i=1}^{\infty} [f(y_{i+1}) - f(y_i)]$ approaches zero with $m(E)$, i.e., with $\sum_{i=1}^{\infty} (y_{i+1} - y_i)$. Thus our theorem is proved.

Uniform absolute continuity.* Consider a sequence of summable functions $f_\mu(x)$ which converge to the function $f(x)$ over the set E.

DEFINITION. The absolute continuity of the integrals

$$\int f_\mu dx$$

is said to be uniform over the set E, if to every positive ϵ there corresponds a δ such that

$$\left| \int_e f_\mu dx \right| < \epsilon$$

independent of μ, provided that e be a portion of E, of measure less than δ (Vitali).

THEOREM *C*.† If the absolute continuity of the integrals

$$\int f_\mu(x)dx$$

* de la Vallee Poussin, Transactions, pp. 445 et seq.

† de la Vallee Poussin, loc. cit.

is uniform over the set E, $f(x)$ is summable over E and

$$\lim_{\mu=\infty} \int_E f_\mu dx = \int_E f(x) dx.$$

The definition of absolute continuity and the proof of the theorem (C) can be extended to the case where the functions f_μ involve more than one variable.

We generalize the notion of uniform absolute continuity as follows:

Definition. Let us suppose that the function $f(x, y, \alpha)$ is summable over the set $E(x, y)$ for all values of α belonging to a set $A(\alpha)$. Then if the integral

$$F(\alpha) = \int_E \int f(x, y, \alpha) dx dy$$

is such that to every positive ϵ there corresponds a δ such that

$$\left| \int_e \int f(x, y, \alpha) dx dy \right| < \epsilon$$

for all values of α in A, provided only that $m(e) < \delta$, the absolute continuity of the integral $F(\alpha)$ is said to be uniform with regard to α.

This definition includes the previous one.

Theorem D. If $u(x, y)$ is limited and summable (linearly) with regard to y for all values of x in the region $[a \leq x \leq b,\ c \leq y \leq d]$ and if u is absolutely continuous in x for all values of y, and if $\partial u / \partial x$ is summable superficially in the same region, then the function

$$F(x, y) = \int_a^y u(x, \eta) d\eta$$

is a continuous function of the two variables x and y. For consider the expression

$$\begin{aligned} F(x, y) - F(x_0, y_0) \\ &= \int_a^y u(x, \eta) d\eta - \int_a^{y_0} u(x, \eta) d\eta + \int_a^{y_0} u(x, \eta) d\eta - \int_a^{y_0} u(x_0, \eta) d\eta \\ &= \int_{y_0}^{y} u(x, \eta) d\eta + \int_a^{y_0} [u(x, \eta) - u(x_0, \eta)] d\eta. \end{aligned}$$

The first term can be made as small as we please by taking $|y - y_0|$ small enough, independent of x, since u is limited. The second term can be written

$$\int_a^{y_0} d\eta \int_{x_0}^{x} \frac{\partial u}{\partial \xi} d\xi = \int_{R(x_0, x;\, a, y_0)} \int \frac{\partial u}{\partial \xi} d\xi d\eta,$$

since u is absolutely continuous in x, and $\partial u/\partial\xi$ is summable superficially. For the latter reason the integral

$$\int_{R(x_0,\, x;\, a,\, y_0)}\int \frac{\partial u}{\partial \xi}\,d\xi d\eta$$

approaches zero with $m(R)$, i.e., with $|x - x_0|$. Hence finally, $F(x, y) - F(x_0, y_0)$ approaches zero with $\sqrt{(x - x_0)^2 + (y - y_0)^2}$, and this is what we wished to prove.

§ 2. I. THEOREM. If (1) $u(x, y)$, $v(x, y)$, are limited and summable (superficially) throughout the region $R(0 \leq x \leq 1, 0 \leq y \leq 1)$,

(2) u, likewise v, is an absolutely continuous function of x for every value of y, and of y for every value of x,

(3) $\partial u/\partial x$, $\partial u/\partial y$, $\partial v/\partial x$, $\partial v/\partial y$, considered where they exist, are summable (superficially) in R,

(4) $f(x, y)$, $g(x, y)$ are summable in R,

(5) *the following equations are satisfied for every rectangle* $D[X_1 \leq x \leq X_2,\ Y_1 \leq y \leq Y_2]$ *inside* R, *for which the left-hand members have a meaning:*

$$(A)\qquad \int_c \frac{\partial u}{\partial x}dy - \frac{\partial u}{\partial y}dx = \int\int_D f(x, y)dxdy,$$

$$(B)\qquad \int_c \frac{\partial v}{\partial x}dy - \frac{\partial v}{\partial y}dx = \int\int_D g(x, y)dxdy,$$

(6) the rectangle $S[a \leq x \leq b,\ c \leq y \leq d]$ is such that the absolute continuity of the integral $\int(\partial u/\partial x)dy$ is uniform in the neighborhood of $x = a$, $x = b$; similarly the absolute continuity of $\int(\partial u/\partial y)dx$ is uniform in the neighborhood of $y = c$, $y = d$,

(7) at nearly every point of the boundary of S, u, $\partial u/\partial x$, $\partial u/\partial y$, v, $\partial v/\partial x$, $\partial v/\partial y$ are the superficial derivatives of their respective double integrals $\int\int u\,dx\,dy$, etc.

then for every such rectangle S

$$(C)\qquad \int_c \left(v\frac{\partial u}{\partial x} - u\frac{\partial v}{\partial x}\right)dy - \left(v\frac{\partial u}{\partial y} - u\frac{\partial v}{\partial y}\right)dx = \int\int_S (vf - ug)dxdy.$$

It is to be noted that, since u, $\partial u/\partial x$, $\partial u/\partial y$, etc., are summable superficially, the values of a, (c), for which the lines $x = a$, $(y = c)$, do not satisfy condition (7), form a set of zero measure.* Also, for the same reason, the values of X_1, X_2, Y_1, Y_2, for which the left-hand members of equations (A), (B) do not have a meaning, form a set of zero measure.† Hence conditions (5) and (7) are satisfied by nearly every rectangle in R.

* A summable function is the derivative of its indefinite integral nearly everywhere.

† Cours d'analyse, vol. II, pp. 117 et seq.

Similarly, since u, v are limited, equation (C) will have a meaning for nearly every rectangle S in R.

II. The proof of the theorem is obtained by the use of approximating polynomials,* defined as follows:

$$\bar{P}_\mu[u(x, y)] = \frac{1}{k_\mu^2}\int\int_R u(\xi, \eta)[1 - (\xi - x^2)]^\mu[1 - (\eta - y)^2]^\mu d\xi d\eta,$$

where

$$k_\mu = 2\int_0^1 (1 - t^2)^\mu dt,$$

$$P_\mu[u(x, y)] = \frac{1}{k_\mu^2}\int_{-x}^{1-x} d\xi \int_{-y}^{1-y} u(\xi + x, \eta + y)[1 - \xi^2]^\mu[1 - \eta^2]^\mu d\eta.$$

Let

$$\bar{P}_\mu[u(x, y)] = \frac{1}{k_\mu^2}\int_{x-\epsilon}^{x+\epsilon} d\xi \int_{y-\epsilon}^{y+\epsilon} u(\xi, \eta)[1 - (\xi - x)^2]^\mu[1 - (\eta - y)^2]^\mu d\eta,$$

where ϵ is a positive number, less than any of the numbers a, $1 - b$, c, $1 - d$. Evidently

$$P_\mu[u(x, y)] = \frac{1}{k_\mu^2}\int_{-\epsilon}^{\epsilon} d\xi \int_{-\epsilon}^{\epsilon} u(\xi + x, \eta + y)[1 - \xi^2]^\mu[1 - \eta^2]^\mu d\eta.$$

It is well known that $P_\mu[u(x, y)]$ approaches $u(x, y)$ nearly everywhere in S, and therefore by condition (7) nearly everywhere on the boundary of S, as $\mu \doteq \infty$. It is to be noted that the functions $P_\mu[u(x, y)]$, $\bar{P}_\mu[u(x, y)]$ are limited in their sets. It is also well known that

$$Q_\mu[u] = P_\mu[u] - \bar{P}_\mu[u]$$

approaches zero and that $\bar{P}_\mu[u]$ approaches u as $\mu \doteq \infty$.

III. The method of proof is as follows: Since $P_\mu[u]$, $P_\mu[v]$ are polynomials, they satisfy the conditions of Green's Theorem, as usually stated, and we can therefore write:

$$(1)\quad \int_c \left\{ P_\mu[v]\frac{\partial P_\mu[u]}{\partial x} - P_\mu[u]\frac{\partial P_\mu[v]}{\partial x} \right\} dy - \left\{ P_\mu[v]\frac{\partial P_\mu[u]}{\partial y} - P_\mu[u]\frac{\partial P_\mu[v]}{\partial y} \right\} dx$$

$$= \int\int_S \left\{ P_\mu[v]\nabla^2 P_\mu[u] - P_\mu[u]\nabla^2 P_\mu[v] \right\} dxdy.$$

It is proved that, on the boundary of S

$$\lim_{\mu=\infty}\frac{\partial P_\mu[u]}{\partial x} = \frac{\partial u}{\partial x}, \qquad \lim_{\mu=\infty}\frac{\partial P_\mu[u]}{\partial y} = \frac{\partial u}{\partial y}$$

* Cours d'analyse, vol. II, pp. 126 et seq.

and that, in S

$$\lim_{\mu \to \infty} \nabla^2 P_\mu[u] = f(x, y)$$

and similar relations are proved for $P_\mu^1[v]$. It is shown moreover that the limits of the above integrals, in (1), are equal to the integrals of the limits of their integrands. We thus obtain, by taking the limits of both sides of (1), the formula to be proved:

$$\int_c \left(\frac{v\,\partial u}{\partial x} - u\frac{\partial v}{\partial x}\right) dy - \left(v\frac{\partial u}{\partial y} - u\frac{\partial v}{\partial y}\right) dx = \int_s\int (vf - ug)dxdy.$$

We have to prove that the absolute continuity of the integrals in (1) is uniform with regard to μ (Theorem C).

IV. We consider now the quantity:

$$Q_\mu[u(x, y)] = P_\mu[u(x, y)] - \overline{P}_\mu[u(x, y)].$$

A glance at the diagram of the regions of integration of $P_\mu[u]$ and $\overline{P}_\mu[u]$ will show that $Q_\mu[u]$ is the sum of eight terms of which the following are typical:

$$\alpha_\mu[x, y] = \frac{1}{k_\mu^{\ 2}}\int_{y+\epsilon}^{1} d\eta \int_{x+\epsilon}^{1} u(\xi, \eta)[1 - (\xi - x^2)]^\mu[1 - (\eta - y)^2]^\mu d\xi,$$

$$\beta_\mu[x, y] = \frac{1}{k_\mu^{\ 2}}\int_{y-\epsilon}^{y+\epsilon} d\eta \int_{x+\epsilon}^{1} u(\xi, \eta)[1 - (\xi - x)^2]^\mu[1 - (\eta - y)^2]^\mu d\xi.$$

With regard to α_μ, we see that the inner integral is an absolutely continuous function of x for all values of η; hence, using Leibnitz' rule, and differentiating with respect to x, we obtain:

$$\frac{\partial\alpha_\mu}{\partial x} = \frac{1}{k_\mu^{\ 2}}\int_{y+\epsilon}^{1} d\eta \int_{x+\epsilon}^{1} u(\xi, \eta)D_x[1 - (\xi - x)^2]^\mu[1 - (\eta - y)^2]^\mu d\xi$$
$$- \frac{1}{k_\mu^{\ 2}}\int_{y+\epsilon}^{1} u(x + \epsilon, \eta)[1 - \epsilon^2]^\mu[1 - (\eta - y)^2]^\mu d\eta.$$

Since the integrand of the second term is an absolutely continuous function of x for all values of η we can differentiate again, using Leibnitz' rule for Lebesgue integrals* and obtain:

$$\frac{\partial^2\alpha_\mu}{\partial x^2} = \frac{1}{k_\mu^{\ 2}}\int_{y+\epsilon}^{1} d\eta \int_{x+\epsilon}^{1} u(\xi, \eta)D_x^2[1 - (\xi - x)^2]^\mu[1 - (\eta - y)^2]^\mu d\eta$$
$$- \frac{1}{k_\mu^{\ 2}}\int_{y+\epsilon}^{1} u(x + \epsilon, \eta)\{D_x[1 - (\xi - x)^2]^\mu\}_{\xi=x+\epsilon}[1 - (\eta - y)^2]^\mu d\eta$$
$$- \frac{1}{k_\mu^{\ 2}}\int_{y+\epsilon}^{1} \frac{\partial}{\partial x} u(x + \epsilon, \eta)[1 - \epsilon^2]^\mu[1 - (\eta - y)^2]^\mu d\eta.$$

* Cours d'analyse, Volume II, p. 123.

The expressions for $\partial\alpha_\mu/\partial y$ and $\partial^2\alpha_\mu/\partial y^2$ are similar to those for $\partial\alpha_\mu/\partial x$, $\partial^2\alpha_\mu/\partial x^2$.

Differentiating $\beta_\mu[u(x, y)]$ we obtain:

$$\frac{\partial\beta_\mu}{\partial x} = \frac{1}{k_\mu{}^2}\int_{y-\epsilon}^{y+\epsilon} d\eta \int_{x+\epsilon}^{1} u(\xi, \eta) D_x[1 - (\xi - x)^2]^\mu [1 - (\eta - y)^2]^\mu d\xi$$
$$- \frac{1}{k_\mu{}^2}\int_{y-\epsilon}^{y+\epsilon} u(x + \epsilon, \eta)[1 - \epsilon^2]^\mu [1 - (\eta - y)^2]^\mu d\eta$$

$$\frac{\partial^2\beta_\mu}{\partial x^2} = \frac{1}{k_\mu{}^2}\int_{y-\epsilon}^{y+\epsilon} d\eta \int_{x+\epsilon}^{1} u(\xi, \eta) D_x{}^2[1 - (\xi - x)^2]^\mu [1 - (\eta - y)^2]^\mu d\xi$$
$$- \frac{1}{k_\mu{}^2}\int_{y-\epsilon}^{y+\epsilon} u(x + \epsilon, \eta) \{D_x[1 - (\xi - x)^2]\}_{\xi = x+\epsilon} [1 - (\eta - y)^2]^\mu d\eta$$
$$- \frac{1}{k_\mu{}^2}\int_{y-\epsilon}^{y+\epsilon} \frac{\partial}{\partial x} u(x + \epsilon, \eta)[1 - \epsilon^2]^\mu [1 - (\eta - y)^2]^\mu d\eta.$$

Writing β_μ in the form

$$\frac{1}{k_\mu{}^2}\int_{x+\epsilon}^{1} d\xi \int_{y-\epsilon}^{y+\epsilon} u(\xi, \eta)[1 - (\xi - x)^2]^\mu [1 - (\eta - y)^2]^\mu d\eta$$

we obtain:

$$\frac{\partial\beta_\mu}{\partial y} = \frac{1}{k_\mu{}^2}\int_{x+\epsilon}^{1} d\xi \int_{y-\epsilon}^{y+\epsilon} u(\xi, \eta)[1 - (\xi - x)^2]^\mu D_y[1 - (\eta - y)^2]^\mu d\eta$$
$$+ \frac{1}{k_\mu{}^2}\int_{x+\epsilon}^{1} u(\xi, y + \epsilon)[1 - (\xi - x)^2]^\mu [1 - \epsilon^2]^\mu d\xi$$
$$- \frac{1}{k_\mu{}^2}\int_{x+\epsilon}^{1} u(\xi, y - \epsilon)[1 - (\xi - x)^2]^\mu [1 - \epsilon^2]^\mu d\xi,$$

$$\frac{\partial^2\beta_\mu}{\partial y^2} = \frac{1}{k_\mu{}^2}\int_{x+\epsilon}^{1} d\xi \int_{y-\epsilon}^{y+\epsilon} u(\xi, \eta)[1 - (\xi - x)^2]^\mu D_y{}^2[1 - (\eta - y)^2]^\mu d\eta$$
$$+ \frac{1}{k_\mu{}^2}\int_{x-\epsilon}^{1} u(\xi, y + \epsilon)[1 - (\xi - x)^2]^\mu \{D_y[1 - (\eta - y)^2]^\mu\}_{\eta = y+\epsilon} d\xi$$
$$- \frac{1}{k_\mu{}^2}\int_{x+\epsilon}^{1} u(\xi, y - \epsilon)[1 - (\xi - x)^2]^\mu \{D_y[1 - (\eta - y)^2]^\mu\}_{\eta = y+\epsilon} d\xi$$
$$+ \frac{1}{k_\mu{}^2}\int_{x+\epsilon}^{1} \frac{\partial}{\partial y} u(\xi, y + \epsilon)[1 - (\xi - x)^2]^\mu [1 - \epsilon^2]^\mu d\xi$$
$$- \frac{1}{k_\mu{}^2}\int_{x+\epsilon}^{1} \frac{\partial}{\partial y} u(\xi, y - \epsilon)[1 - (\xi - x)^2]^\mu [1 - \epsilon^2]^\mu d\xi.$$

V. From the expressions for $\partial\alpha_\mu/\partial x$, $\partial\alpha_\mu/\partial y$, $\partial\beta_\mu/\partial x$, $\partial\beta_\mu/\partial y$ we can now deduce the fact that $(\partial/\partial x)Q_\mu[u]$ is an absolutely continuous function

of x, and that $(\partial/\partial y)Q_\mu[u]$ is absolutely continuous in y. For, consider $\partial\alpha_\mu/\partial x$. The first term is of the same character as α_μ itself, which is the sum of terms of the type

$$Ax^iy^j\int_{x+\epsilon}^1 d\xi\int_{y+\epsilon}^1 \xi^r\eta^s u(\xi,\eta)d\eta,$$

which is obviously, absolutely continuous since u is limited. The second term satisfies the conditions of Theorem B, and is therefore an absolutely continuous function of x, for all values of y. The same kind of reasoning can be applied to the discussion of $\partial\alpha_\mu/\partial y$, $\partial\alpha_\mu/\partial x$, $\partial\beta_\mu/\partial y$ and we can infer therefore that $(\partial/\partial x)Q_\mu[u]$, $(\partial/\partial y)Q_\mu[u]$ are absolutely continuous functions, respectively, of x and y.

VI. We have next to show that $(\partial^2/\partial x^2)Q_\mu[u]$ and $(\partial^2/\partial y^2)Q_\mu[u]$ are summable, superficially, in the rectangle S. It will be seen that the only terms in the expressions representing $\partial^2\alpha_\mu/\partial x^2$, $\partial^2\alpha_\mu/\partial y^2$, $\partial^2\beta_\mu/\partial x^2$, $\partial^2\beta_\mu/\partial y^2$ whose summability is not evident are of a form of which the following is typical:

$$\frac{1-\epsilon^2}{k_\mu^2}\int_{x+\epsilon}^1\frac{\partial}{\partial y}u(\xi,y-\epsilon)[1-(\xi-x)^2]^\mu d\xi.$$

The presence of the factor $[1-(\xi-x)^2]^\mu$ in the integrand will evidently not affect the discussion. We shall therefore consider the function

$$F(x,y)=\int_{x+\epsilon}^1\frac{\partial}{\partial y}u(\xi,y-\epsilon)d\xi=\frac{\partial}{\partial y}\int_{x+\epsilon}^1 u(\xi,y-\epsilon)d\xi$$

and change the variable of integration by putting

$$\xi=x+\epsilon+(1-x-\epsilon)t.$$

We thus obtain

$$F(x,y)=\int_0^1\frac{\partial}{\partial y}u[x+\epsilon+(1-x-\epsilon)t,\,y-\epsilon][1-x-\epsilon]dt.$$

Consider now the function $(\partial/\partial Y)u(X,Y)$ which by hypothesis, is summable in the square $[0\leq X\leq 1,\ 0\leq Y\leq 1]$. If we define the function

$$u(X,Y,T)=u(X,Y)$$

for all values of T, it is evident that $(\partial/\partial Y)u[X,Y,T]$ is summable in the three-dimensional region $[0\leq X\leq 1,\ 0\leq Y\leq 1,\ 0\leq T\leq 1]$. It is therefore summable in any measurable subregion of this cube, for instance in the region $E(X,Y,T)$ bounded by the planes

$$T=a,\ T=b;\ Y=c-\epsilon,\ Y=d-\epsilon;\ X=T+\epsilon,\ X=1.$$

Let us now perform the following transformation: using the results of Theorem A

$$X = x + \epsilon + (1 - \epsilon - x)t, \qquad J = \begin{vmatrix} 1 - t & 0 & 1 - \epsilon - x \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix}$$
$$Y = y - \epsilon,$$
$$T = x,$$

$$|J| = 1 - \epsilon - x.$$

$(\partial/\partial Y)u[X, Y, T] = (\partial/\partial Y)u[X, Y]$ thus becomes

$$\frac{\partial}{\partial Y} \cdot u[x + \epsilon + (1 - \epsilon - x)t, y - \epsilon]$$

and the integral,

$$\int_{E(X, Y, T)} \int \int \frac{\partial}{\partial y} u[XYT]dXdYdT$$
$$= \int_{t=0}^{1} \int_{y=c}^{d} \int_{x=a}^{b} \frac{\partial}{\partial y} u[x + \epsilon + (1 - \epsilon - x)t, y - \epsilon][1 - \epsilon - x]dxdydt.$$

We have now shown that $(\partial/\partial y)u[x+\epsilon+(1-\epsilon-x)t, y-\epsilon][1-\epsilon-x]$ is summable in the region corresponding to $E(X, Y, T)$, i.e., in $[a \leq x \leq b, c \leq y \leq d, 0 \leq t \leq 1]$. We therefore know that $F(x, y)$ is summable in S,* and we can write, in fact

$$\int_S \int F(x, y)dxdy$$
$$= \int_S \int dxdy \int_0^1 \frac{\partial}{\partial y} u[x + \epsilon + (1 - \epsilon - x)t, y - \epsilon][1 - \epsilon - x]dt.$$

VII. We next show that the absolute continuity of

$$\int_S \int \Delta^2 Q_\mu dxdy$$

is uniform with regard to μ. The term

$$\frac{(1 - \epsilon^2)^\mu}{k_\mu^2} \int_{x+\epsilon}^{1} \frac{\partial}{\partial y} u(\xi, y - \epsilon)[1 - (\xi - x)^2]^\mu d\xi$$

is typical of those which go to make up $\Delta^2 Q_\mu[u]$. Ignoring the factor $(1 - \epsilon^2)^\mu/k_\mu^2$ which approaches zero as $\mu \doteq \infty$, and writing

$$\int \int \int \pi_\mu(x, y, t)dtdxdy$$
$$= \int_c^d \int_a^b \int_0^1 \frac{\partial}{\partial y} u[x + \epsilon + (1 - \epsilon - x)t, y - \epsilon][1 - \epsilon - x]$$
$$\times [1 - \{\epsilon + (1 - \epsilon - x)t\}^2]^\mu dtdxdy$$

* Cours d'analyse, volume II, pp. 117 et seq.

we see that

$$\pi_\mu(x, y, t) = h(x, y, t)[1 - \{\epsilon + (1 - \epsilon - x)t\}^2]^\mu$$

and that $|\pi_\mu(x, y, t)| \leq |h(x, y, t)|$, where

$$h(x, y, t) = (\partial/\partial y)u[x + \epsilon + (1 - \epsilon - x)t, y - \epsilon][1 - \epsilon - x].$$

We have shown that $h(x, y, t)$ is summable in the region $S'[a \leq x \leq b, c \leq y \leq d, 0 \leq t \leq 1]$. Hence $|h(x, y, t)|$ is summable in S' and π_μ the product of two summable functions, one of which is limited, is also summable in S', therefore

$$\int_S\int dxdy \int_0^1 \pi_\mu(x, y, t)dt \leq \int_S\int dxdy \int_0^1 |h(x, y, t)|dt.$$

Since h is independent of μ we see from this equation that the absolute continuity of the integrals

$$\int\int dxdy \int_0^1 \pi_\mu[x, y, t]dt$$

is uniform is S, and our statement is proved.

VIII. We need to show that $(\partial/\partial x)Q_\mu[u]$ and $(\partial/\partial y)Q_\mu[u]$ are continuous functions of the two variables x and y. It will be sufficient to consider the quantity $\partial\alpha_\mu/\partial x$.

The first term of $\partial\alpha_\mu/\partial x$ is like α_μ itself, which is a sum of terms of the type

$$Ax^iy^j \int_{y+\epsilon}^1 d\eta \int_{x+\epsilon}^1 \xi^r\eta^s u(\xi, \eta)d\xi,$$

which is continuous if the integral itself is continuous. But the integral has a limited integrand and is evidently a continuous function of (x, y).

The second term of $\partial\alpha_\mu/\partial x$ is a sum of terms of the type

$$By^i \int_{y+\epsilon}^1 \eta^r u(x + \epsilon, \eta)d\eta,$$

which is a continuous function of x and y if the integral is itself continuous. But the integral is of the type considered in Theorem D, and is therefore continuous. We have now proved that:

$\partial Q_\mu/\partial x$, $\partial Q_\mu/\partial y$ are continuous functions of x and y in S,

$\partial Q_\mu/\partial x$ is an absolutely continuous function of x in S,

$\partial Q_\mu/\partial y$ is an absolutely continuous function of y in S.

$\nabla^2 Q_\mu$ is summable in S.

In fact, if we remember that ϵ is less than a, $1 - b$, c, $1 - d$, it becomes evident that the above statements apply also to the rectangle $\bar{S}[a - \epsilon' \leq x \leq b + \epsilon', c - \epsilon' \leq y \leq d + \epsilon']$, where ϵ' is a small positive

constant such that

$$\epsilon + \epsilon' < a,\ 1 - b,\ c,\ 1 - d.$$

We can now apply Green's Formula,* and obtain for any square s inside of $\bar{S}$, such that s contains the point (x_0, y_0).

$$\int_c \frac{\partial Q_\mu}{\partial x} dy - \frac{\partial Q_\mu}{\partial y} dx = \int_s \int \nabla^2 Q_\mu dxdy.$$

Therefore, since $\nabla^2 Q_\mu dxdy$ is summable,

$$\lim_{m(s)=0} \frac{1}{m(s)} \int_c \frac{\partial Q_\mu}{\partial x} dy - \frac{\partial Q_\mu}{\partial y} dx = \nabla^2 Q_\mu[u(x_0, y_0)]$$

for nearly every point (x_0, y_0) of $\bar{S}$, consequently of S.

IX. Consider now the expression

$$\frac{\partial}{\partial x} \bar{P}_\mu[u(x, y)] = \frac{1}{k_\mu{}^2} \int_{-\epsilon}^{\epsilon} d\eta \int_{-\epsilon}^{\epsilon} \frac{\partial}{\partial x} u(\xi + x, \eta + y)[1 - \xi^2]^\mu[1 - \eta^2]^\mu d\xi.$$

It is easy to show by means of Theorem A, as we applied it to the function $(\partial/\partial y)u[x + \epsilon + (1 - \epsilon - x)t, y - \epsilon]$, that the function

$$\frac{\partial}{\partial x} u(\xi + x, \eta + y)$$

is summable in the three-dimensional region

$$[-\epsilon \leq \xi \leq \epsilon,\ a \leq x \leq b,\ -\epsilon \leq \eta \leq \epsilon].$$

Hence we can write, treating $\partial u/\partial y$ in the same way,

$$\begin{aligned}
&\frac{1}{\sigma} \int_c \frac{\partial}{\partial x} \bar{P}\mu[u] dy - \frac{\partial}{\partial y} \bar{P}\mu[u] dx \\
&\quad = \frac{1}{k_\mu{}^2} \int_{-\epsilon}^{\epsilon} d\eta \int_{-\epsilon}^{\epsilon} [1 - \xi^2]^\mu [1 - \eta^2]^\mu \frac{1}{\sigma} \int_c \frac{\partial u(\xi + x, \eta + y)}{\partial x} dy \\
&\qquad\qquad - \frac{\partial u(\xi + x, \eta + y)}{\partial y} dx \\
&\quad = \frac{1}{k_\mu{}^2} \int_{-\epsilon}^{\epsilon} d\eta \int_{-\epsilon}^{\epsilon} [1 - \xi^2]^\mu [1 - \eta^2]^\mu \frac{1}{\sigma} \int_\sigma \int f(\xi + x, \eta + y) dxdy,
\end{aligned}$$

where σ is the area of a square containing the point $(\xi + x_0, \eta + y_0)$ as center. Let us choose a sequence of numbers σ_i approaching zero as a limit as $\mu \doteq \infty$. Then, if the absolute continuity of the integrals

$$\int_{-\epsilon}^{\epsilon} d\xi \int_{-\epsilon}^{\epsilon} d\eta (1 - \xi^2)^\mu (1 - \eta^2)^\mu \frac{1}{\sigma_i} \int_{\sigma_i} \int f(\xi + x, \eta + y) dxdy$$

* Cours d'analyse, II, p. 124.

is uniform with regard to i over the rectangle $[-\epsilon \leqq \xi \leqq \epsilon, -\epsilon \leqq \eta \leqq \epsilon]$, we can write

$$\lim_{\sigma=0}\frac{1}{\sigma}\int \frac{\partial}{\partial x}\bar{P}_\mu[u]dy - \frac{\partial}{\partial y}\bar{P}_\mu[u]dx$$

$$= \frac{1}{k_\mu^2}\int_{-\epsilon}^{\epsilon} d\eta \int_{-\epsilon}^{\epsilon}(1-\xi^2)^\mu(1-\eta^2)^\mu f(x_0+\xi, y_0+\eta)d\xi.$$

To prove that such is the case we can ignore the factor $[1-\xi^2]^\mu[1-\eta^2]^\mu$, which is positive and limited, and consider the integral

$$\int_{-\epsilon}^{\epsilon} d\xi \int_{-\epsilon}^{\epsilon} d\eta \frac{1}{\sigma_i}\int_{\sigma_i}\int f(\xi+x, \eta+y)dxdy$$

$$= \int_{R(\epsilon,\sigma)}\int\int\int \frac{1}{\sigma_i} f(\xi+x, \eta+y)d\xi d\eta dx dy,$$

since, as is easily shown by Theorem A, $f(\xi+x, \eta+y)$ is summable in the four-dimensional region

$$R[\epsilon, \sigma] \text{ or } \left[-\epsilon \leqq \xi \leqq \epsilon, -\epsilon \leqq \eta \leqq \epsilon,\right.$$

$$\left. x_0 - \frac{\sqrt{\sigma_i}}{2} \leqq x \leqq x_0 + \frac{\sqrt{\sigma_i}}{2}, y_0 - \frac{\sqrt{\sigma_i}}{2} \leqq y \leqq y_0 + \frac{\sqrt{\sigma_i}}{2}\right].$$

The last integral can be written

$$\int_{x_0-\frac{\sqrt{\sigma_i}}{2}}^{x_0+\frac{\sqrt{\sigma_i}}{2}} dx \int_{y_0-\frac{\sqrt{\sigma_i}}{2}}^{y_0+\frac{\sqrt{\sigma_i}}{2}} dy \frac{1}{\sigma_i}\int_{-\epsilon}^{\epsilon}\int_{-\epsilon}^{\epsilon} f(\xi+x, \eta+y)d\xi d\eta.$$

Now, since f is summable two-dimensionally in the fundamental region $R(0 \leqq X \leqq 1, 0 \leqq Y \leqq 1)$, by the definition of absolute continuity of an integral,

$$\int_{e(\xi,\eta)}\int f(\xi+x, \eta+y)d\xi d\eta$$

approaches zero with the measure of $e(\xi, \eta)$ (which is a part of the rectangle $(-\epsilon \leqq \xi \leqq \epsilon, -\epsilon \leqq \eta \leqq \epsilon)$ *independent of* x *and* y, provided only that (x, y) lie in the rectangle S. Hence

$$\int_{y_0-\frac{\sqrt{\sigma_i}}{2}}^{y_0+\frac{\sqrt{\sigma_i}}{2}} dy \int_{x_0-\frac{\sqrt{\sigma_i}}{2}}^{x_0+\frac{\sqrt{\sigma_i}}{2}} dx \frac{1}{\sigma_i}\int_{e(\xi,\eta)}\int f(\xi+x, \eta+y)d\xi d\eta$$

$$\leqq \int_{y_0-\frac{\sqrt{\sigma_i}}{2}}^{y_0+\frac{\sqrt{\sigma_i}}{2}} dy \int_{x_0-\frac{\sqrt{\sigma_i}}{2}}^{x_0+\frac{\sqrt{\sigma_i}}{2}} dx \cdot \frac{1}{\sigma_i}\cdot\omega$$

$$\leqq \omega,$$

where ω approaches zero with the measure of $e(\xi, \eta)$ independently of x, y. Hence

$$\lim_{\sigma=0} \frac{1}{\sigma} \int_{c(\sigma)} \frac{\partial}{\partial x} \bar{P}_\mu[u]dy - \frac{\partial}{\partial y} \bar{P}_\mu[u]dx$$
$$= \frac{1}{k_\mu^2} \int_{-\epsilon}^{\epsilon} d\eta \int_{-\epsilon}^{\epsilon} [1 - \xi^2]^\mu [1 - \eta^2]^\mu f(x_0 + \xi, y_0 + \eta) d\xi.$$

But we have proved that

$$\lim_{\sigma=0} \frac{1}{\sigma} \int_{c(\sigma)} \frac{\partial}{\partial x} \bar{Q}_\mu[u]dy - \frac{\partial}{\partial y} Q_\mu[u]dx = \nabla^2 Q_\mu[u(x_0, y_0)].$$

Hence by addition

$$\lim_{\sigma=0} \frac{1}{\sigma} \int_{c(\sigma)} \frac{\partial}{\partial x} P_\mu[u]dy - \frac{\partial}{\partial y} P_\mu[u]dx$$
$$= \nabla^2 Q_\mu[u(x_0 y_0)] + \frac{1}{k_\mu^2} \int_{-\epsilon}^{\epsilon} d\eta \int_{-\epsilon}^{\epsilon} (1 - \xi^2)^\mu (1 - \eta^2)^\mu f(x_0 + \xi, y_0 + \eta) d\xi.$$

Therefore, since f is summable and $\lim_{\mu=\infty} \nabla^2 Q_\mu[u(x_0, y_0)] = 0$,

$$\lim_{\mu=\infty} \nabla^2 P_\mu[u(x_0, y_0)] = f(x_0, y_0)$$

at nearly every point (x_0, y_0) of the region S.

Similarly we can write

$$\lim_{\mu=\infty} \nabla^2 P_\mu[v(x_0, y_0)] = g(x_0, y_0).$$

X. We now have to discuss the absolute continuity of the integrals appearing in equation (1), with regard to uniformity.

The functions $P_\mu[u]$, $P_\mu[v]$ are limited in their sets: they will therefore not affect the discussion. We therefore consider the integral

$$\int_c^d \left[\frac{\partial}{\partial x} P[u(x, y)]\right]_{x=a} dy = \int_c^d \left[\frac{\partial}{\partial x} \bar{P}_\mu[u(x, y)]\right]_{x=a} dy$$
$$+ \int_c^d \left[\frac{\partial}{\partial x} Q_\mu[u(x, y)]\right]_{x=a} dy,$$

for which we have to prove the uniform absolute continuity.

I say that the quantities $\partial\alpha_\mu/\partial x$, $\partial\beta_\mu/\partial x$ which are representative of those which make up $\partial Q_\mu/\partial x$ are limited in their sets. For the integrands in the expression for $\partial\alpha_\mu/\partial x$ both contain the factor $[1 - (\eta - y)^2]^\mu \gtrless (1 - \epsilon^2)^\mu$ and since $(1/k_\mu^2)(1 - \epsilon^2)^\mu$ approaches zero as $\mu \doteq \infty$, u being limited, it is evident that the functions $[\partial\alpha_\mu/\partial x]$ are limited in their

set. Moreover, the integrands in the expression for $\partial\beta_\mu/\partial x$ contain the factors $D_x[1-(\xi-x)^2]^\mu$ and $(1-\epsilon^2)^\mu$ respectively. But since $D_x[1-(\xi-x)^2]^\mu$ is an infinitesimal of the order of $\mu[1-(\xi-x)^2]^{\mu-1} \lessapprox \mu(1-\epsilon^2)^{\mu-1}$ and since

$$\frac{\mu(1-\epsilon^2)^{\mu-1}}{k_\mu{}^2} \doteq 0$$

as $\mu = \infty$, we see that the functions $[\partial\beta_\mu/\partial x]$ are limited in their set. It follows therefore that the functions $[\partial Q_\mu/\partial x]$ are limited in their set, and obviously approach zero everywhere in D.

Hence

$$\lim_{\mu=\infty}\int_c^d\left[\frac{\partial Q_\mu[u(x,y)]}{\partial x}\right]_{x=a}dy = 0$$

and

$$\lim_{\mu=\infty}\int_c^d\left[\frac{\partial}{\partial x}P_\mu[u(x,y)]\right]_{x=a}dy = \lim_{\mu=\infty}\int_c^d\left[\frac{\partial}{\partial x}\bar{P}_\mu[u(x,y)]\right]_{x=a}dy.$$

We have now to show that the absolute continuity of the integral

$$\int_c^d\left[\frac{\partial}{\partial x}\bar{P}_\mu[u(x,y)]\right]_{x=a}dy$$

is uniform with regard to μ. Now

$$\int_c^d\left[\frac{\partial}{\partial x}\bar{P}_\mu[u(x,y)]\right]_{x=a}dy$$

$$= \int_c^d dy\frac{1}{k_\mu{}^2}\int_{-\epsilon}^{\epsilon}\int_{-\epsilon}^{\epsilon}\left[\frac{\partial}{\partial y}u(\xi+x,\eta+y)\right]_{x=a}(1-\xi^2)^\mu(1-\eta^2)^\mu d\xi d\eta.$$

By means of Theorem A, it is easy to show, by the method of VI that $[(\partial/\partial x)u(\xi+x,\eta+y)]$ is summable in the three-dimensional region

$$[-\epsilon\leq\xi\leq\epsilon,\ -\epsilon\leq\eta\leq\epsilon,\ c\leq y\leq d],$$

we can therefore write

$$\int_c^d\left[\frac{\partial}{\partial x}\bar{P}_\mu[u(x,y)]\right]_{x=a}dy$$

$$= \frac{1}{k_\mu{}^2}\int_{-\epsilon}^{\epsilon}d\xi\int_{-\epsilon}^{\epsilon}d\eta[1-\xi^2]^\mu[1-\eta^2]^\mu\int_c^d\left[\frac{\partial}{\partial x}u(\xi+x,\eta+y)\right]_{x=a}dy.$$

We now make use of the statement (6) of our hypothesis, that the absolute continuity of the integral

$$\int\frac{\partial}{\partial x}u(x,y)dy$$

is uniform in the neighborhood of $x = a$. So far we have not made use of the fact that the constant ϵ can be chosen as small as we please. We now choose ϵ so small that the absolute continuity of $\int (\partial/\partial x) u(x, y) dy$ will be uniform for all values of x in the interval $a - \epsilon \leq x \leq a + \epsilon$.

Then

$$\left| \int_{e(y)} \left[\frac{\partial}{\partial x} \bar{P}_\mu [u(x, y)] \right]_{x=a} dy \right| \leq \left| \frac{1}{k_\mu^2} \int_{-\epsilon}^{\epsilon} d\xi \int_{-\epsilon}^{\epsilon} d\eta (1 - \xi^2)^\mu (1 - \eta^2)^\mu \omega \right|,$$

where $\omega \doteq 0$ with $m[e(y)]$ independent of ξ and η. Thus our statement is proved.

XI. With regard to the right-hand member of equation (1) we proceed as follows:

We have shown that

$$\nabla^2 P_\mu[u] = \nabla^2 Q_\mu[u] + \frac{1}{k_\mu^2} \int_{-\epsilon}^{\epsilon} d\eta \int_{-\epsilon}^{\epsilon} (1 - \xi^2)^\mu (1 - \eta^2)^\mu f(x_0 + \xi, y_0 + \eta) d\xi$$

and we have shown that the absolute continuity of $\int\int \nabla^2 Q_\mu[u] dxdy$ is uniform in S. We have now to show that the absolute continuity of

$$\int\int \bar{P}_\mu[f(x, y)] dxdy$$
$$= \frac{1}{k_\mu^2} \int\int dxdy \int_{-\epsilon}^{\epsilon} \int_{-\epsilon}^{\epsilon} f(x + \xi, y + \eta)(1 - \xi^2)^\mu (1 - \eta^2)^\mu d\xi d\eta$$

is uniform in S.

Since, as we have stated before, $f(x + \xi, \eta + y)$ is summable in the four-dimensional region

$$[a \leq x \leq b, c \leq y \leq d, -\epsilon \leq \xi \leq \epsilon, -\epsilon \leq \eta \leq \epsilon],$$

we can write

$$\int_S \int \bar{P}_\mu[f(x, y)] dxdy$$
$$= \frac{1}{k_\mu^2} \int_{-\epsilon}^{\epsilon} d\xi \int_{-\epsilon}^{\epsilon} d\eta (1 - \xi^2)^\mu (1 - \eta^2)^\mu \int_S \int f(x + \xi, y + \eta) dxdy.$$

But since $f(x, y)$ is summable in R $[0 \leq x \leq 1, 0 \leq y \leq 1]$ by the definition of absolute continuity, since $\epsilon < a, 1 - b, c, 1 - d$, the absolute continuity of

$$\int\int f(x + \xi, y + \eta) dxdy$$

is uniform for all values of the parameters ξ, η in the region

$$[-\epsilon \leq \xi \leq \epsilon, -\epsilon \leq \eta \leq \epsilon].$$

Hence

$$\int_{e(x,y)}\int \overline{P}_\mu[f(x, y)]dxdy \leq \frac{1}{k_\mu^{\,2}}\int_{-\epsilon}^{\epsilon} d\xi \int_{-\epsilon}^{\epsilon} \omega[1 - \xi^2]^\mu[1 - \eta^2]^\mu d\eta \leq \omega,$$

where $\omega \doteq 0$ with $m[e(x, y)]$ independent of ξ and η. The proof is now complete; for in view of the nature of the rectangle S, we know that $(\partial/\partial x)P_\mu[u]$, $(\partial/\partial y)P_\mu[u]$ approach $\partial u/\partial x$, $\partial u/\partial y$ nearly everywhere along its boundary.

Moreover $\nabla^2 P_\mu[u]$ approaches $f(x, y)$ nearly everywhere in D. Similar statements apply to $(\partial/\partial x)P_\mu[v]$, $(\partial/\partial y)P_\mu[v]$ and to $\nabla^2 P_\mu[v]$. Taking the limits of both sides of equation (1) as $\mu \doteq \infty$ we obtain finally

$$\int_c \left(v\frac{\partial u}{\partial x} - u\frac{\partial v}{\partial x}\right)dy - \left(v\frac{\partial u}{\partial y} - u\frac{\partial v}{\partial y}\right)dx = \int_S\int (vf - ug)dxdy.$$

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BILINEAR OPERATIONS GENERATING ALL OPERATIONS RATIONAL IN A DOMAIN Ω.

BY NORBERT WIENER.

The notion of a corpus, or domain of rationality, is familiar to mathematicians. It can easily be extended to cover sets, not of numbers, but of functions of one or more variables (which may, indeed, degenerate into numbers), exhibiting the group property with reference to the four fundamental operations—addition, subtraction, multiplication, and division—barring only division by 0. Now, it should be noted that these four operations are not only methods of combining functions, but are themselves functions of two variables. Thus $x + y$ is a function of x and y. To combine two functions $f(x)$ and $g(x)$ by addition is to eliminate u and v from the three equations

$$z = u + v,$$
$$u = f(x),$$
$$v = g(x),$$

and thus to define z in terms of x.

It is thus possible to regard a function-corpus containing addition, subtraction, multiplication, and division, as a set Σ of functions or operations which contains a sub-set Σ' such that every combination of operations of Σ by operations of Σ' belongs to Σ. In certain cases, Σ' and Σ will coincide. Thus let Σ be the set of all rational functions with coefficients rational in a domain Ω: if $F(x_1, x_2, \cdots, x_n)$, $f_1(y_{11}, y_{12}, \cdots, y_{1k})$, $f_2(y_{21}, y_{22}, \cdots, y_{2l})$, $\cdots, f_n(y_{n1}, y_{n2}, \cdots, y_{nm})$ belong to Σ, $F(f_1, f_2, \cdots, f_n)$ will also belong to Σ, and this whether the y's are all distinct or not. A set of operations, whether a functional corpus or not, which possesses this invariancy under iteration, may be called an *iterative field*.

Iterative fields constitute one of the generalizations of substitution-groups—in fact, substitution-groups are iterative fields of functions of one variable. Now, in a group it is often possible to pick out a very restricted set of elements—a so-called basis—which by their combination generate all the elements of the group. Iterative fields likewise have bases—addition, multiplication, subtraction, and division constitute a basis for the iterative field consisting of all rational operations with rational coefficients. It thus becomes a matter of some interest to determine

the smallest possible basis of an iterative field—in particular to determine if and how the field can be generated by the iteration of a single operation. If this is possible, it corresponds to a cyclical group.

In this paper the question at issue is how an iterative field consisting of all rational operations with their coefficients rational in a domain Ω may be generated by the iteration of a single operation of the form

$$x \mid y = \frac{A + Bx + Cy + Dxy}{E + Fx + Gy + Hxy}.$$

The chief result obtained is that if Ω is the domain of rationals, the necessary and sufficient condition that $x \mid y$ constitute a basis is that there is a linear transformation T such that $T^{-1}\{T(x) \mid T(y)\}$ or $T^{-1}\{T(y) \mid T(x)\}$ is of the form

$$\frac{x - y}{x + Ay},$$

where A is any rational number, or else of the form

$$\frac{n(x - y + xy)}{ny + xy},$$

when n is any integer other than 0. Thus $(x - y)/(3x + 6y)$ and $(xy + 3x - 3y - 2)/(xy + x + 3y + 3)$ both serve as bases for all rational operations with rational coefficients, since the first is of the form

$$\frac{1}{3}\left(\frac{3x - 3y}{3x + 2(3y)}\right),$$

and is a transform of

$$\frac{x - y}{x + 2y};$$

while the second may be written

$$\frac{2((x + 1) - (y + 1) + (x + 1)(y + 1))}{2(y + 1) + (x + 1)(y + 1)} - 1,$$

and is a transform of

$$\frac{2(x - y + xy)}{2y + xy}.$$

1. **Definitions.** A *bilinear* operation is a binary operation of the form

$$\frac{A + Bx + Cy + Dxy}{E + Fx + Gy + Hxy}.$$

An operation *rational in a domain* Ω is a rational operation whose coefficients belong to Ω.

An operation $f(x_1, x_2, \cdots, x_k)$ is said to *generate* a class K of operations and numbers which may be considered as operations on no variables, when it has the following four properties.

(1) K contains f.

(2) If K contains $g(x_1, x_2, \cdots, x_{i-1}, x_i, x_{i+1}, \cdots, x_{j-1}, x_j, x_{j+1}, \cdots, x_k)$, K contains $g(x_1, x_2, \cdots, x_{i-1}, x_i, x_{i+1}, \cdots, x_{j-1}, x_i, x_{j+1}, \cdots, x_k)$, and also $g(x_1, x_2, \cdots, x_{i-1}, x_j, x_{i+1}, \cdots, x_{j-1}, x_i, x_{j+1}, \cdots, x_k)$.

(3) If K contains $g(x_1, x_2, \cdots, x_k)$ and $h(y_1, y_2, \cdots, y_l)$, K contains the operation on $x_1, x_2, \cdots, x_{i-1}, x_{i+1}, \cdots, x_k, y_1, y_2, \cdots, y_l$ obtained by substituting h for x_i.

(4) K contains no proper sub-set K' with properties (1), (2), and (3).

Infinity will be considered a proper argument for an operation, and $f(\infty)$ will be defined as $\lim\limits_{x=\infty} f(x)$, if this latter expression has a unique significance. Infinity will be considered as a member of every domain.

If T is the non-singular linear transformation

$$x' = \frac{a + bx}{c + dx},$$

the *transform* of $f(x_1, x_2, \cdots, x_k)$ by T is defined as the operation which expresses y in terms of $x_1, x_2, \cdots, x_k$, and is expressed in implicit form in the following set of simultaneous equations.

$$y' = f(x_1', x_2', \cdots, x_k'),$$

$$y' = \frac{a + by}{c + dy},$$

$$x_i' = \frac{a + bx_i}{c + dx_i} \qquad (1 \leq i \leq k).$$

2. **Theorem.*** *Every transform of $(x - y)/(x + Gy)$ by a T rational in $\Omega_{(G)}$ generates all operations rational in $\Omega_{(G)}$ if only $G \neq -1$.*

Proof. For the sake of brevity, we shall represent the operation $(x - y)/(x + Gy)$ by the symbol $x|y$. If we can show that whenever $G \neq -1$, we can build up a chain of definitions wherein addition, subtraction, multiplication, division, and G are derived from iterations of $x|y$, it is clear that every operation rational in $\Omega_{(G)}$ can be derived from $x|y$ by iteration. The formation of *ad hoc* definitions to cover all cases where the iterations given become indeterminate offers no difficulty.

There are two cases. First, let $G \neq \pm 1$. Our chain of definitions will read as follows.

* In this and the following theorems, the operations generated by the bilinear operations in question may be undefined for a set of isolated values of their arguments.

$$
\begin{aligned}
0 &= x|x, \\
1 &= x|0, \\
x|y &= 1|(y|x), \\
xy &= x|(1|y), \\
A(x) &= 1|[(x|1)|1] \text{ [in ordinary symbolism, } A(x) = (G + 1 - x], \\
x - y &= \{xA[(y|x)A(0)]\}|A(0), \\
x + y &= x - [(a - y) - a], \\
G &= [a - 1|(0|x)] - a.
\end{aligned}
$$

If $G = 1$, this chain is replaced by the following sequence of definitions:

$$
\begin{aligned}
0 &= x|x, \\
1 &= x|0, \\
-1 &= 0|x, \\
x|y &= 1|(y|x), \\
xy &= x|(1|y), \\
A(x) &= x^2|\{[x|(1|x)]|[(x|1)\cdot(x^2|1)]\} \text{ [i.e., } 1|(2x - 1)], \\
B(x) &= x^2|\{\{(-1)A\{(1|x^2)[(x|1)|(-1|x)]\}\}|A\{(x|1)|(-1|x)\}\} \\
&\qquad \left[\text{i.e., } 1 - \frac{x}{y}\right], \\
x - y &= xB(y|x), \\
x + y &= x - (-1)y.
\end{aligned}
$$

Since $x|y$ is rational in $\Omega_{(G)}$, it only generates operations rational in $\Omega_{(G)}$, while we have just seen that it generates all operations rational in $\Omega_{(G)}$. A transformation T rational in $\Omega_{(G)}$ is simply a permutation of the numbers and operations of $\Omega_{(G)}$ among themselves. Hence the T-transform of $x|y$ generates all operations rational in $\Omega_{(G)}$, and no others.

3. **Theorem.** *Every transform of*

$$\frac{n(x - y + xy)}{ny + xy}$$

by a T rational in $\Omega_{(1)}$ generates all operations rational in $\Omega_{(1)}$, if n is a non-0 integer in $\Omega_{(1)}$.

Proof. The discussion of the last theorem applies *in toto*, except that the chair of definitions must be altered. We shall denote

$$\frac{n(x - y + xy)}{ny + xy}$$

by $x|y$. Three cases arise.

(1) $n = 1$. Consider the following definitions:

$$
\begin{aligned}
0 &= (x|x)|x, \\
-1 &= 0|x, \\
\infty &= x|0, \\
-x &= \infty|(x|-1),
\end{aligned}
$$

$$
\begin{aligned}
A(x, y) &= -\{-(-x \mid -x) \mid -[(-y \mid -y) \mid (-y \mid -y)]\} \left[\text{i.e., } 1 - \frac{x}{y}\right],\\
1 &= \infty \mid \infty,\\
x/y &= A\{A(x, y), 1\},\\
xy &= x/(1/y),\\
x - y &= xA(y, x),\\
x + y &= x - [(a - y) - a].
\end{aligned}
$$

(2) $n > 1$. Consider the following definitions:

$$
\begin{aligned}
a^1(x) &= x \mid x,\\
a^{k+1}(x) &= a^1[a^k(x)] \text{ [i.e., } a^k(x) = nx/(kx + n)],\\
0 &= a^n(x) \mid x,\\
-1 &= 0 \mid x,\\
\infty &= x \mid 0,\\
P(x, y) &= a^{n-1}(x) \mid a^{n-1}(y) \text{ [i.e., } (x - y + xy)/(y + xy)],\\
-x &= \infty \mid (x \mid -1),\\
A(x, y) &= -P\{-P(-x, -x), -P[P(-y, -y), P(-y, -y)]\}\\
&\qquad \text{[as before]},\\
1 &= P(\infty, \infty),\\
x/y &= A\{A(x, y), 1\},\\
xy &= x/(1 \mid y),\\
x - y &= xA(y, x),\\
x + y &= x - [(a - y) - a].
\end{aligned}
$$

(3) n is negative. $x \mid y$ is a transform by $y' = y/(y + 1)$ of $-n(y - x + xy)/(-nx + xy)$, which differs from the cases considered in (1) and (2) merely by the interchange of x and y.

4. **Theorem.** *If $x \mid y$ is a bilinear operation generating every operation rational in any domain, it is a transform by a linear transformation rational in some $\Omega_{(G)}$ either of*

$$\frac{x - y}{x + \alpha y} \text{ or of } \frac{y - x}{y + \alpha x},$$

where α is a number in $\Omega_{(G)}$ other than ∞ or -1, or of

$$\frac{n(x - y + xy)}{ny + xy},$$

where n is a number in $\Omega_{(G)}$ other than ∞ or 0.

Proof. Let

$$x \mid y = \frac{A + Bx + Cy + Dxy}{E + Fx + Gy + Hxy}.$$

Consider the roots of the cubic equation $x \mid x = x$. If for any of these roots, r_1, r_2, or r_3, $r_i \mid r_i$ is determinate, then no sequence of iterations of $\mid$ on r_i can generate anything but r_i. As the condition we have supposed

is equivalent to the assumption that $x|y$ be continuous when x and y are in the neighborhood of r_i, it follows that when x and y are in the neighborhood of r_i, $x|y$ is also in that neighborhood. It results that if $f(x_1, x_2, \cdots, x_n)$ is an operation derived by the iteration of $x|y$, and if $x_1, x_2, \cdots, x_n$ are all in the neighborhood of r_i, $f(x_1, x_2, \cdots, x_n)$ will also be in that neighborhood. This restricts f to a relatively narrow range of variation. In particular, it cannot be any constant other than r_i itself. Since the four fundamental operations $+$, $-$, $\times$, and $\div$ generate all rational constants, it follows that $x|y$ cannot generate these, and hence cannot generate all rational operations with rational coefficients.

Consequently every root of $x|x = x$, be it finite or infinite, is also a finite or infinite root of both the quadratics,

$$Hx^2 + (F + G)x + E = 0, \tag{1}$$

and

$$Dx^2 + (B + C)x + A = 0. \tag{2}$$

If these two equations are not equivalent, the algorithm of the greatest common factor leads to the common root

$$r = -\frac{\begin{vmatrix} E & H \\ A & D \end{vmatrix}}{\begin{vmatrix} F+G & H \\ B+C & D \end{vmatrix}}.$$

This is rational in the domain of the coefficients, and is furthermore, from what has been said, the only finite or infinite root of $x|x = x$. This we shall call case A.

If (1) and (2) are equivalent, it may be seen on inspection that each of their roots is also a root of $x|x = x$, which is *in extenso*

$$Hx^3 + (F + G - D)x^2 + (E - B - C)x - A = 0. \tag{3}$$

This will then have the root $r_1 = D/H$, which is rational in the domain of the coefficients. It follows from what has been said that r_1 is a root of (1). The other root of (1) is $r_2 = (E/H)/(D/H)$, or E/D, which is also rational in the domain of the coefficients. The roots of (3) are then the double root r_1 and the single root r_2. This we shall call case B.

We have seen that a linear transform of an operation has essentially the same iterative properties as its original. Accordingly we may subject the $x|y$ of case A to the transformation $u' = u + r$, thus obtaining the operation

$$x \| y = \frac{A' + B'x + C'y + D'xy}{E' + F'x + G'y + H'xy}.$$

We shall now have 0 as a root of the quadratic

(1′) $$H'x^2 + (F' + G')x + E',$$

and a triple root of

(3′) $$H'x^3 + (F' + G' - D')x^2 + CE' - B' - c')x - A' = 0.$$

It follows that $A' = E' = B' + C' = F' + G' - D' = 0$, whence

$$x \,\|\, y = \frac{B'x - B'y + (F' + G')xy}{F'x + G'y + H'xy}.$$

The transformation $u' = B'u/[F'u + (F' + G')]$ reduces this to the form*

$$x \,|||\, y = \frac{n(x - y + xy)}{ny + xy}.$$

Exceptional cases might seem to arise where $B' = 0$ or $F' + G' = 0$. If $B' = 0$, on transforming $x \,\|\, y$ by $u' = 1/u$, we get

$$x \,|||\, y = G'x + F'y + H',$$

and it is manifest that the iteration of this can only lead to linear operations. If $F' + G' = 0$, $x \,\|\, y$ reduces to the form

$$\frac{x - y}{P(x - y) + Qxy}.$$

If we transform this by $u' = u/(Pu + Q)$ we get

$$x \,|||\, y = \frac{x - y}{xy}.$$

Be it noted that the degree of every term in the numerator is odd, while the power of every term in the denominator is even. It is easy to show that if f and g be two rational operations each in the form either of the quotient of a polynomial whose terms are all of odd degree by a polynomial whose terms are all of even degree, or of the quotient of a polynomial whose terms are all of even degree by a polynomial whose terms are all of odd degree, the operation obtained by substituting f as one of the arguments of g will share in this property. Thus if $f(x, y) = (x - y)/xy$ and $g(x, y, z) = (x^2 + y^2 + z^2)/(x + yz^2)$, $g(x, y, f(u, v))$ will be $(x^4y^2 + x^2y^4 + x^2 - 2xy + y^2)/(x^3y^2 + x^2y - 2xy^2 + y^3)$, where all the terms of the numerator are of even degree and all the terms in the denominator are of odd degree. Hence the iteration of an operation in this class can never lead beyond the confines of the class, and must consequently fail to generate all operations rational in any domain.

* $n \neq 0$, as the expression would otherwise reduce to a constant. $n \neq \infty$; otherwise, whatever x, $x \,|||\, x = x$.

In case B, submit $x \| y$ to the transformation $u' = (r_2u - r_1)/(u - 1)$. $x \| y$ will go into an operation

$$x \| y = \frac{A' + B'x + C'y + D'xy}{E' + F'x + G'y + H'xy}.$$

The double root of $x \| x = x$, or of

$$H'x^3 + (F' + G' - D')x^2 + (E' - B' - C')x - A' = 0,$$

will be 0, while the single root will be ∞. Hence

$$A' = H' = E' - B' - C' = 0.$$

Since 0 and ∞ are also roots of

$$H'x^2 + (F' + G')x + E' = 0,$$

and of

$$D'x^2 + (B' + C')x + A' = 0,$$

D' and E' also vanish. From these facts concerning the coefficients, it may be seen that $x \| y$ reduces to

$$\frac{x - y}{Mx + Ny}.$$

Transforming this operation by $u' = u/M$, the result is of the form

$$\frac{x - y}{x + \alpha y}.$$

If $M = 0$, the above transformation is impossible, but if we transform by $u' = -u/N$, we get $(y - x)/y$, of the form $(y - x)/(y + \alpha x)$.

We have thus actually carried through the transformations which reduce $x \| y$ to one of the standard forms, and have found them to be rational in the domain of the coefficients of $x \| y$.

5. **Theorem.** *If a transform of $n(x - y + xy)/(ny + xy)$ generates all operations rational in any domain, then n is of the form p/q, where p is an integer in $\Omega_{(n)}$ and q is a natural number sharing with p no numerical factor, while there is some natural number k such that q is a factor of p^k.* (An example of the situation contemplated in this theorem is $p = 1 + \sqrt{-5}$; $q = 2$; $k = 2$; $p^k = -4 + 2\sqrt{-5} = q(-2 + \sqrt{-5})$.)

Proof. Consider the class of all numbers in $\Omega_{(n)}$ of the form

$$\frac{p^k + aq}{p^k + bq},$$

where q and k are natural numbers, a and b are integers in $\Omega_{(n)}$, and p is an integer in $\Omega_{(n)}$ sharing no numerical factor with q. Suppose that every number in $\Omega_{(n)}$ can be represented in this form. Let r/s be any number in

$\Omega_{(n)}$ and let r and s be integers in $\Omega_{(n)}$. Then

$$(r - s)p^k = qt,$$

where t is an integer in $\Omega_{(n)}$, namely $sa - rb$. By properly choosing r and s, $r - s$ may be made to contain no factor, actual or ideal, in common with q. Hence, if all numbers in $\Omega_{(n)}$ are reducible to the above form, for some value of k, q is a factor of p^k.

Now, if $n = p/q$, and

$$x|y = \frac{n(x - y + xy)}{ny + xy}$$

generates all operations rational in $\Omega_{(n)}$, $(p^k + aq)/(p^k + bq)$ must assume all values in $\Omega_{(n)}$. For if we let x and y be $(p^k + aq)/(p^k + bq)$ and $(p^l + \alpha q)/(p^l + \beta q)$, it follows that

$$\begin{aligned} x|y &= \frac{(p/q)(x - y + xy)}{(p/q)y + xy} \\ &= \frac{\dfrac{p}{q}\left\{\dfrac{p^k + aq}{p^k + bq} - \dfrac{p^l + \alpha q}{p^l + \beta q} + \dfrac{p^k + aq}{p^k + bq}\cdot\dfrac{p^l + \alpha q}{p^l + \beta q}\right\}}{\dfrac{p}{q}\cdot\dfrac{p^l + \alpha q}{p^l + \beta q} + \dfrac{p^k + aq}{p^k + bq}\cdot\dfrac{p^l + \alpha q}{p^l + \beta q}} \\ &= \frac{p^{k+l+1} + Aq}{p^{k+l+1} + Bq}, \end{aligned}$$

where A and B are integers in $\Omega_{(n)}$. Hence if K is the class of all members of $\Omega_{(n)}$ of the form $(p^k + aq)/(p^k + bq)$, when x and y belong to this class, $x|y$ will also belong to this class. If $x|y$ is to generate all operations rational in $\Omega_{(n)}$, and *a fortiori* all numbers in $\Omega_{(n)}$, K must coincide with $\Omega_{(n)}$. In the previous paragraph we have seen this condition to be equivalent to that expressed in the formulation of the theorem.

6. **Theorem.** *The necessary and sufficient condition that the set of all operations generated by the rational bilinear operation $x|y$ be the set of all rational operations with rational coefficients is that $x|y$ be a rational transform of*

$$\frac{x - y}{x + Ay} \quad \textit{or} \quad \frac{y - x}{y + Ax},$$

where A is any rational number other than -1, or of

$$\frac{n(x - y + xy)}{ny + xy},$$

where n is any integer other than 0.

Proof. This results immediately from 2–5, if the domain in question be made that of the rationals.

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ON THE ENUMERATION OF PROPER AND IMPROPER REPRESENTATIONS IN HOMOGENEOUS FORMS.

By E. T. Bell.

1. By the usual definition, a particular representation of n in the form

$$\Sigma a_{ij} x_i x_j, \ (i, j = 1, 2, \cdots, r), \tag{1}$$

e.g., through $(x_1, x_2, \cdots, x_r) = (x_1', x_2', \cdots, x_r')$, is proper or improper according as the G.C.D. of the x' is $\geqq 1$; and two representations $(x_1', x_2', \cdots, x_r')$, $(x_1'', x_2'', \cdots, x_r'')$ are identical only when

$$x_i' = x_i'', \ (i = 1, \cdots, r).$$

Let $T(n)$, $P(n)$ denote respectively the total number of representations, and the number of proper representations of n in (1). Then, the Σ referring to every positive d such that n/d^2 is an integer, we have* directly from the definitions

$$T(n) = \Sigma P(n/d^2). \tag{2}$$

Similarly, if in place of (1) we have a homogeneous form of degree s, there is between the corresponding T, P the relation

$$T(n) = \Sigma P(n/d^s), \tag{3}$$

the Σ extending to all positive d such that n/d^s is an integer. Practically nothing of importance, except in the case of binary cubics, being known concerning T, P when $s > 2$, we shall confine the discussion to (2), merely indicating in § 15 the nature of the easy extension whereby all of the general formulæ for $s = 2$ can be carried up to $s > 2$ whenever specific theorems for the latter cases shall be available.

2. If the strictly arithmetical theory, due principally to Eisenstein, H. J. S. Smith and Minkowski, be used to find $T(n)$, $P(n)$, the natural (and historical) order appears to be first the determination of $P(n)$, and thence by (2) the deduction of $T(n)$. If, however we seek $T(n)$, $P(n)$ algebraically, either by elliptic functions or otherwise (cf. § 9), $T(n)$ always appears first, $P(n)$ entering, if at all, only through cumbersome and artificial transformations of the analysis appropriate to $T(n)$. It

* For a detailed discussion of a particular case, cf. Bachmann, *Die Arithmetik der Quadratischen Formen*, p. 602. Note that in accordance with modern usage we have not assumed $a_{ij} = a_{ji}$ in (1).

seems in fact that without previous knowledge of the results to be attained, suitable transformations of the fundamental identities would present themselves but seldom. On the other hand, it has frequently been pointed out* that where solutions of arithmetical problems by elliptic functions or other algebraic means exist, they are in general much simpler, shorter, and less delicate in application than the corresponding investigations by processes peculiar to the theory of numbers. Hence it is of some importance to invert the present problem, deducing the $P(n)$ directly from the $T(n)$, the latter in many instances being given very simply by algebraical methods. It will be seen in §§ 6, 7 how this may always be done, and how, in §§ 8–11, for one of the most important classes of representations the general formulæ assume simple and interesting shapes. By means of the formulæ developed the deduction of $P(n)$ from $T(n)$ is immediate; in §§ 12–14 a few illustrations are given in the derivation of $P(n)$ theorems due to Eisenstein and Liouville, the proofs for which seem not to have been published hitherto; also in writing down some new results for 6, 8, 10 and 12 squares. But the object of this paper being the general method, and not special consequences, applications are included only in sufficient number to make clear the use of the formulæ. It may be mentioned that by means of the formulæ in §§ 10, 11 all of Liouville's numerous $P(n)$ theorems which he published without proofs in the second series of his Journal, may be demonstrated almost at a glance. His $T(n)$ theorems were all proved by Pepin, cf. § 9; the proofs of the rest, all of which have been found by the methods of this paper, will appear in the Journal de Mathematiques for 1919. Theorems such as those of Liouville and Eisenstein concerning special quadratic forms are of importance as guides in the general theory of (1), which still is far from complete.

3. All letters m, n, d, δ denote positive non-zero integers; *the m's are always odd, and the n's arbitrary.* We define n to be simple if it is divisible by no square > 1; and adopting Sylvester's convenient term, call the number of distinct prime factors of n its multiplicity. Consider

$$(4) \qquad F(1) = 1, \qquad F(n_1 n_2) = F(n_1)F(n_2), \qquad D(n_1, n_2) = 1,$$

where $D(n_1, n_2)$ is the G.C.D. of n_1, n_2. Functions $F, f, g, \cdots$ satisfying (4) we shall call factorable. If by the nature of the function, factorable $f(n)$ is undefined for $n = 1$, then by convention $f(1) = 1$; also, $f(x) = 0$ when x is not a positive integer. We shall require Möbius' $\mu(n)$, which $= 0$ if n is not simple, and which otherwise $= +1$ or -1 according as the multiplicity of n is even or odd. It is readily seen that $\mu(n)$ is factor-

* See, for example, Glaisher's remarks in the Proc. London Math. Society (2), 5 (1907), pp. 489–490, § 16.

able, and that

$$(5) \qquad \sum_n \mu(d) = 0, \; n > 1,$$

the notation $\sum_n$ indicating that the sum is taken with respect to all divisors d of n. The fundamental property (5) is well known; nevertheless we recall one of the shortest ways by which it may be established, as the same applies to all subsequent identities concerning factorable functions. Since $\mu(n)$ is factorable, it suffices to verify (5) for $n = p^a$, where p is prime. The divisors d in this case are $1, p, p^2, \cdots, p^a$; and from the definition of $\mu(n)$: $\mu(1) = 1$; $\mu(p) = -1$; $\mu(p^a) = 0$, $a > 1$. In the same way each of the factorable function identities given later may be proved by verifying them for $n = p^a$ directly from the definitions of the particular functions involved. These verifications, presenting no difficulty or interest, will be omitted.

4. Let (d, δ) denote any pair of conjugate divisors of n, so that $n = d\delta$. Form the value of $\varphi_1(x)\psi(y)$ for $(x, y) = (d, \delta)$, sum $\varphi_1(d)\psi(\delta)$ over all pairs (d, δ), and denote the result by $\sum_n \varphi_1(d)\psi(\delta)$. Let (δ_1, δ_2) denote any pair of conjugate divisors of δ, so that

$$n = d\delta, \qquad \delta = \delta_1\delta_2, \qquad n = d\delta_1\delta_2.$$

Put $\psi(n) = \sum_n \varphi_2(d)\varphi_3(\delta)$; whence,

$$\sum_n \varphi_1(d)\psi(\delta) = \sum_n [\varphi_1(d) \sum_\delta \varphi_2(\delta_1)\varphi_3(\delta_2)] = \sum_n \varphi_1(d_1)\varphi_2(d_2)\varphi_3(d_3),$$

the last summation extending to all triads (d_1, d_2, d_3) such that $n = d_1d_2d_3$. With this notation, and (p, q, r), (i, j, k) any permutations of $(1, 2, 3)$, the following is obvious:

$$\sum_n [\varphi_p(d) \sum_\delta \varphi_q(\delta_1)\varphi_r(\delta_2)] = \sum_n \varphi_i(d_1)\varphi_j(d_2)\varphi_k(d_3).$$

Clearly this may be extended to any number of functions $\varphi_1, \varphi_2, \cdots, \varphi_r$. It will be found that the case $r = 4$ is required in some of the verifications:

$$\sum_n [\sum_d \varphi_1(d_1)\varphi_2(d_2) \sum_\delta \varphi_3(\delta_1)\varphi_4(\delta_2)] = \sum_n \varphi_1(d_1)\varphi_2(d_2)\varphi_3(d_3)\varphi_4(d_4),$$

in the first of which $\sum_n, \sum_d, \sum_\delta$ refer to all (d, δ), (d_1, d_2), (δ_1, δ_2) respectively such that $n = d\delta$, $d = d_1d_2$, $\delta = \delta_1\delta_2$; and $\sum_n$ in the second to all tetrads (d_1, d_2, d_3, d_4) such that $n = d_1d_2d_3d_4$. Also it is evident that

$$\sum_n [\sum_d \varphi_p(d_1)\varphi_q(d_2) \sum_\delta \varphi_r(\delta_1)\varphi_s(\delta_2)] = \sum_n [\varphi_i(d) \sum_\delta \varphi_j(\delta_1)\varphi_k(\delta_2)\varphi_l(\delta_3)],$$

where in the second, $\sum_n, \sum_\delta$ refer respectively to all (d, δ) such that $n = d\delta$,

and to all $(\delta_1, \delta_2, \delta_3)$ such that $\delta = \delta_1\delta_2\delta_3$; and (p, q, r, s), (i, j, k, l) are any permutations of $(1, 2, 3, 4)$. By the method of § 3 it is easily shown that if $\varphi_1, \varphi_2, \varphi_3, \varphi_4$ are factorable, then each of the multiple sums in this paragraph is a factorable function of n.

5. It is known that if $g_1(n)$, $g_2(n)$, $g_3(n)$ are factorable, then a factorable $f(n)$ exists such that

$$\sum_n f(d)g_1(\delta) = \sum_n g_2(d)g_3(\delta). \tag{6}$$

Moreover, if $f(n)$, $g(n)$ both satisfy (6), then $f(n) = g(n)$ for all n; viz., (6) has a unique factorable solution. The following special case is of importance presently. Write $u_r(n) \equiv n^r$; then clearly $u_r(n)$ is a factorable function of n, and (5) may be written in the form

$$\sum_n \mu(d)u_0(\delta) = 0, \qquad n > 1.$$

Hence, if $g(n)$ is given and factorable, a unique factorable $f(n)$ may always be found such that

$$\sum_1 f(d)g(\delta) = 1; \qquad \sum_n f(d)g(\delta) = 0,\ n > 1.$$

It suffices to observe that $f(n)$ is uniquely determined by

$$\sum_n f(d)g(\delta) = \sum_n \mu(d)u_0(\delta).$$

6. Returning to § 1, let $\epsilon(n) = 1$ or 0 according as n is or is not a square. Clearly $\epsilon(n)$ is factorable. Replace (2) by its equivalent,

$$T(n) = \sum_n \epsilon(d)P(\delta); \tag{7}$$

and hence, for δ any divisor of n,

$$T(\delta) = \sum_\delta \epsilon(\delta_1)P(\delta_2). \tag{8}$$

Multiply (8) throughout by $f(d)$, where d is the conjugate of δ, and sum with respect to all pairs (d, δ). By § 4 the result may be written

$$\sum_n T(d)f(\delta) = \sum_n [P(d) \sum_\delta \epsilon(\delta_1)f(\delta_2)]. \tag{9}$$

Hence if $f(n)$ is determined as the solution of

$$\sum_n f(d)\epsilon(\delta) = \sum_n u_0(d)\mu(\delta), \tag{10}$$

we shall have by the last of § 5 the following unique expression for $P(n)$ in terms of $T(n)$:

$$P(n) = \sum_n T(d)f(\delta). \tag{11}$$

Write $\mu(n) \times \mu(n) \equiv \mu^2(n)$; then $\mu(n)$ being factorable, so also by the last of § 4 is $\sum_n \mu(d)\mu^2(\delta)$, since $\mu^2(n)$ obviously is factorable; and as in § 3

it may be verified without difficulty that the factorable function of n,

$$\sum_{n} [\epsilon(d) \sum_{\delta} \mu(\delta_1)\mu^2(\delta_2)] = 1 \text{ or } 0$$

according as $n = 1$ or $n > 1$; that is,

$$f(n) = \sum_{n} \mu(d)\mu^2(\delta) \tag{12}$$

is the required solution of (10). Finally, then the inversion (11) of (2) may be written

$$P(n) = \sum_{n} [T(d) \sum_{\delta} \mu(\delta_1)\mu^2(\delta_2)]. \tag{13}$$

7. To reduce (11), (13) to forms whose arithmetical significance is immediate, consider $f(n)$, which henceforth shall denote the function defined in (12). As in § 3, on verifying the statement for p^a, it is clear that $f(n)$ vanishes unless n is the square of a simple number, when the value is $+1$ or -1 according as the multiplicity of $\sqrt{n}$, or what is the same thing, the multiplicity of n, is even or odd. Hence (11), (13) may be paraphrased: The number of proper representations of n in (1) is equal to the sum of the total numbers of representations in (1) of all those divisors of n whose conjugates are squares of simple numbers of even multiplicity, diminished by the sum of the total numbers of representations in (1) of all those divisors of n whose conjugates are squares of simple numbers of odd multiplicity. It follows that the second sum never exceeds the first.

8. By (12) or § 7, for p prime we have

$$f(p) = 0; \qquad f(p^2) = -1; \qquad f(p^a) = 0,\ a > 2. \tag{14}$$

Combined with $f(1) = 1$, (14) has several important consequences which we proceed to develop. Let $\gamma(x)$ denote any function of x which vanishes when x is not a positive integer; and note that until further restricted, $\gamma(x)$ is not necessarily factorable. Call $\Gamma(n)$ defined by

$$\Gamma(n) = \sum_{n} \gamma(d)f(\delta) \tag{15}$$

the conjugate of $\gamma(n)$. *We shall always denote the conjugate of a given function by capitalizing*; thus the conjugates of $g(n)$, $\zeta_r(n)$, $\xi_r'(n)$, $\cdots$ are $G(n)$, $Z_r(n)$, $\Xi_r'(n)$, $\cdots$ respectively. Consider $\Gamma(p^a)$, where p is prime. By (15),

$$\Gamma(p^a) = \gamma(1)f(p^a) + \gamma(p)f(p^{a-1}) + \cdots + \gamma(p^{a-1})f(p) + \gamma(p^a)f(1); \tag{16}$$

and hence from (14),

$$\Gamma(p) = \gamma(p); \qquad \Gamma(p^a) = \gamma(p^a) - \gamma(p^{a-2}),\ a > 1; \tag{17}$$

both of which are included in the second, without the condition $a > 1$,

since $\gamma(p^{-1}) = 0$. Again, if n is prime to p,

$$\Gamma(p^a n) = \sum_n [\sum_{s=0}^{a} \gamma(p^s d) f(p^{a-s}\delta)]$$
$$= \sum_n [\{\gamma(d)f(p^a) + \gamma(pd)f(p^{a-1}) + \cdots + \gamma(p^{a-1}d)f(p) + \gamma(p^a d)f(1)\}f(\delta)],$$

the last on noticing that p^β and δ being relatively prime and f factorable, $f(p^\beta\delta) = f(p^\beta)f(\delta)$. Whence, on applying (14),

$$(18) \qquad \Gamma(p^a n) = \sum_n [\{\gamma(p^a d) - \gamma(p^{a-2}d)\}f(\delta)],$$

for p prime and not a divisor of n. By repeated application of (18) we get a remarkable symbolic form of the inversion (11). Let

$$n = p^a q^b \cdots r^c \equiv \Pi p^a$$

be the resolution of n into its prime factors; and let

$$\Pi[\gamma(p^a) - \gamma(p^{a-2})], \qquad \Pi'[\gamma(p^a) - \gamma(p^{a-2})]$$

denote respectively the ordinary product

$$[\gamma(p^a) - \gamma(p^{a-2})][\gamma(q^b) - \gamma(q^{b-2})] \cdots [\gamma(r^c) - \gamma(r^{c-2})],$$

and the like taken symbolically as follows: After distribution, each term in Π', such for example as $\gamma(p^a)\gamma(q^b) \cdots \gamma(r^{c-2})$, is to be replaced by the γ of the product of the several arguments of the γ's in that term, e.g., the particular term selected is to be replaced by $\gamma(p^a q^b \cdots r^{c-2})$; and the signs are to be as determined by the formal multiplication. Then from (18) we infer, on putting $n = q^b n_1$ where n_1 is prime to q, reapplying (18) and continuing thus until all the distinct prime powers $q^b, \cdots, r^c$ are exhausted,

$$(19) \qquad n = \Pi p^a, \qquad \Gamma(n) = \Pi'[\gamma(p^a) - \gamma(p^{a-2})].$$

The complete induction is immediate from (18), and need not be written out. Hence* from (11), (15), (19),

$$(20) \qquad n = \Pi p^a, \qquad P(n) = \Pi'[T(p^a) - T(p^{a-2})].$$

9. The special cases of (19), (20) in which $\gamma(n)$, $T(n)$ are factorable have a particular interest and importance. The $T(n)$ may be divided into two classes according as they are or are not factorable. The first includes all of the classical theorems of Gauss, Jacobi, Eisenstein, H. J. S. Smith and Liouville concerning representations of numbers as sums of

* Either (20) or its equivalent in § 7 may be proved from the definitions of $T(n)$, $P(n)$ by what H. J. S. Smith (Papers, I, p. 36) called the principle of cross-classification. The same principle gives also (5) and (19).

2, 4, 6, 8, 10 or 12 squares, with the exception of two additional theorems stated by Liouville, and proved by Glaisher, using elliptic functions, concerning 10 and 12 squares; it includes also the theory of those quadratic forms other than sums of squares to which the processes of elliptic functions are naturally adapted. Also when $T(n)$ is factorable, $T(n)$, $P(n)$ may both be calculated in finite form directly from the real divisors of n alone, without the invention of other functions, always more or less complicated, depending upon the representation of numbers in the given quadratic form in one of lower order. For forms other than sums of squares, the first three factorable $P(n)$, $T(n)$ theorems were stated by Eisenstein in his famous memoir, *Neue Theoreme der höheren Arithmetik* (*Crelle*, 35 (1847), p. 134); but the great mass of known results in this direction is due to Liouville, cf. § 12, footnote. It may be shown* that Liouville's 'formules générales' are equivalent to elementary identities in elliptic functions, whence they follow by a simple method of paraphrase. Hence all of Liouville's $P(n)$, $T(n)$ theorems ultimately depend upon the elements of elliptic functions, Pepin† having deduced the $T(n)$ results from the formules générales, and the $P(n)$ being consequences of these, as we shall presently indicate. The relation of Eisenstein's results to elliptic functions will be glanced at in § 14, footnote. It seems, in short, as lately suggested by Mordell,‡ that elliptic functions may have played a greater part in the discovery of many theorems than has been commonly supposed. In speaking of his $P(n)$ results, Liouville remarks (*J. des Math.* (2), 7 (1862), p. 16), "Il y a du reste à ce sujet, une méthod générale qui s'offre d'elle-même." Since the solution (12) is unique, (11) or its equivalent (13) must be what Liouville had in mind. We shall now examine the factorable case in some detail; (11), (20) apply to any case, factorable or not.

10. For $G(n)$ the conjugate of factorable $g(n)$, and for $T(n)$ factorable, we have from (19), (20),

$$n = \Pi p^a, \qquad G(n) = \Pi[\,g(p^a) - g(p^{a-2})]; \tag{21}$$

$$n = \Pi p^a, \qquad P(n) = \Pi[T(p^a) - T(p^{a-2})]. \tag{22}$$

For many forms (1) it has been found necessary or convenient to distinguish several $T(n)$ according to special factors of n; thus§ for m prime to 3, and

$$n = 2^a 3^\beta m = x^2 + y^2 + z^2 + 3u^2$$

* In a series of papers presented to the American Math. Society, Oct., 1918, Liouville's general formulæ are derived incidentally.

† Journal des Math., 1890, pp. 1–64.

‡ Quarterly Journal Math., 48 (1917–1918), No. 189.

§ For the details cf. Liouville, J. des Math. (2), 8 (1863), pp. 105–114; 193–204.

the $T(n)$ take different forms (which, however, may all be included in one general formula), according as $\alpha = 0$, $\alpha = 1$, $\alpha > 1$, and $\beta = 0$, $\beta = 1$, $\beta > 1$. We say that $T(n)$ for this form has special characters with respect to the primes 2, 3. Let $p_1, p_2, \cdots, p_r$ be the primes with respect to which, for a given form, $T(n)$ has special characters. Then $P(n)$ is most readily investigated either by (18) or by

$$n = n_2 \prod_1^r p_i^{a_i}, \qquad P(n) = P(n_2) \prod_1^r [T(p_i^{a_i}) - T(p_i^{a_i-2})],$$

wherein n_2 is prime to p_i $(i = 1, \cdots, r)$. Examples will be found in the illustrations. The importance of (21) is that for $T(n)$ factorable, $P(n)$ is the conjugate of a factorable function; hence we require such conjugates for (1).

11. A few $g(n)$ occur repeatedly in determinations of $T(n)$ for (1). All those in the literature are included in the following, or in simple modifications of them which it is unnecessary to consider here. Hence we shall find their corresponding $G(n)$, the notation being that of § 10, in order that the necessary data for writing down the corresponding $P(n)$ by (21), (22) may be readily accessible. By the usual convention the value of $(a|b)$, the Legendre-Jacobi symbol, is zero if a, b are not relatively prime, and $(a|b)$ is non-existent when b is even. In the following list the Π-notation has the same significance as in § 8; the Π-forms of the $g(n)$ are immediately evident from the definitions of the specific $g(n)$; and the deduction from these of the corresponding conjugates will be sufficiently clear from the full derivation of one of them. We recall that m is positive and odd. That all of the functions except λ are factorable is clear from their definitions.

(i) Let l denote an odd positive or negative constant integer prime to m, and define the ω, ω' functions by

$$\omega_r(m, l) = \sum_m (d|l)d^r, \qquad \bar{\omega}_r(l, m) = \sum_m (l|d)d^r,$$

$$\omega_r'(m, l) = \sum_m (\delta|l)d^r, \qquad \bar{\omega}_r'(l, m) = \sum_m (l|\delta)d^r.$$

Hence for $m = \Pi p^a$, we have:

$$\omega_r'(m, l) = \Pi\left[\frac{p^{r(a+1)} - (p^{a+1}|l)}{p - (p|l)}\right]; \qquad \bar{\omega}_r'(l, m) = \Pi\left[\frac{p^{r(a+1)} - (l|p^{a+1})}{p^r - (l|p)}\right];$$

and it is easily seen that

$$\omega_r(m, l) = (m|l)\omega_r'(m, l); \qquad \bar{\omega}_r(l, m) = (l|m)\bar{\omega}_r'(l, m).$$

Observing that $(p^{a+1}|l) = (p^{a-1}|l)$, and hence

$$\omega_r'(p^a, l) - \omega_r'(p^{a-2}, l) = \frac{p^{r(a+1)} - p^{r(a-1)}}{p^r - (p|l)} \equiv p^{ra}\left[1 + (p|l)\frac{1}{p^r}\right],$$

we have from (21) for the conjugate of $\omega_r'(m, l)$,

$$\Omega_r'(m, l) \equiv \Pi[\omega_r'(p^a, l) - \omega_r'(p^{a-2}, l)] = m^r\Pi\left[1 + (p|l)\frac{1}{p^r}\right].$$

Any conjugate may be found in the same way. We get thus

$$\bar{\Omega}_r'(l, m) = m^r\Pi\left[1 + (l|p)\frac{1}{p^r}\right] \equiv m^r\Pi\left[1 + (-1)^{\frac{1}{4}(l-1)(p-1)}(p|l)\frac{1}{p^r}\right],$$

the last on using the extended law of quadratic reciprocity; and

$$\Omega_r(m, l) = (m|l)\Omega_r'(m, l); \qquad \bar{\Omega}_r(l, m) = (l|m)\bar{\Omega}_r'(l, m).$$

The first form of $\bar{\Omega}_r'(l, m)$ presents itself directly in the consideration of (1); the second is better adapted to computation, and is equivalent to that occurring (for special values of r, l) in the writings of Eisenstein, Liouville and others. With these are two companions for the even case:

(ii) $$\omega_r(m) = \sum_m (2|d)d^r, \qquad \omega_r'(m) = \sum_m (2|\delta)d^r.$$

From these definitions we find as above,

$$\Omega_r'(m) = (2|m)\Omega_r(m) = m^r\Pi\left[1 + (2|p)\frac{1}{p^r}\right].$$

From the definitions it is clear that $\omega_r(m)$ is the sum of the rth powers of all those divisors of m which are of either form $8k \pm 1$, diminished by the like sum for the divisors of either form $8k \pm 3$; $\omega_r'(m)$ is the sum of the rth powers of all those divisors whose conjugates are of either form $8k \pm 1$, diminished by the like sum for the divisors whose conjugates are of either form $8k \pm 3$. Similarly, *for l prime*, $\omega_r(m, l)$ is the sum of the rth powers of all those divisors of m that are quadratic residues of l, diminished by the like sum for the divisors that are quadratic non-residues; $\bar{\omega}_r(l, m)$ is the sum of the rth powers of all those divisors of m of which l is a quadratic residue, diminished by the like sum for the divisors of which l is a quadratic non-residue; and $\omega_r'(m, l)$, $\bar{\omega}_r'(l, m)$ are the corresponding functions in which the divisors are segregated into classes according to the quadratic characters of their conjugates.

(iii) For $l = -1$ the ω-functions take important forms which, as they occur so frequently, are denoted by special letters. These appear first in the cases when (1) degenerates to a sum of squares; thus, they are familiar through the investigations of Jacobi, Eisenstein, H. J. S. Smith and Glaisher for 2, 6, 10, 14 and 18 squares; they also enter when (1) is a sum of 3, 7 or 11 squares.*

* In a paper presented to the American Mathematical Society, April, 1919, the $T(n)$ are given in the form of finite sums of the functions defined in § 11 when (1) is a sum of 3, 5, 7, 9, 11 or 13 squares, the arguments of the functions forming recurring series of the second order, and the number of odd squares in the representations being either pre-assigned or arbitrary.

Write

$$\bar{\omega}_r(-1, m) \equiv \xi_r(m), \qquad \bar{\omega}_r'(-1, m) \equiv \xi_r'(m);$$

whence

$$\xi_r'(m) = (-1)^{(m-1)/2}\xi_r(m), \qquad \Xi_r(m) = (-1)^{(m-1)/2}\Xi_r'(m),$$

$$\Xi_r'(m) = m^r\Pi\left[1 + (-1|p)\frac{1}{p^r}\right] = m^r\Pi\left[1 + (-1)^{(p-1)/2}\frac{1}{p^r}\right];$$

and $\xi_r(m)$ is the excess of the sum of the rth powers of all those divisors of m that are of the form $4k + 1$ over the like sum for the divisors of the form $4k - 1$; $\xi_r'(m)$ is the similar function in which the conjugates of the divisors are of the respective forms $4k + 1, 4k - 1$. For $n = 2^a m$ we have

$$\xi_r(n) = \xi_r(m), \qquad \xi_r'(n) = 2^{ar}\xi_r'(m);$$

and noticing that by an obvious extension $\omega_r(n, l)$, $\omega_r'(n, l)$ may be defined for $\alpha > 0$, the following conjugates:

$$\Xi_r(2m) = \Xi_r(m); \qquad \Xi_r(2^a m) = 0, \alpha > 1;$$

$$\Xi_r'(2m) = 2^r\Xi_r'(m); \qquad \Xi_r'(2^a m) = 2^{(a-2)r}(2^{2r} - 1)\Xi_r'(m), \alpha > 1.$$

(iv) For $n = 2^a m$, $\alpha \geqq 0$, and $n = \Pi p^a$ the resolution of n into prime factors, $\zeta_r(n)$, $\zeta_r'(n)$ the respective sums of all, of the odd divisors of n, we have in the same way:

$$\zeta_r(n) = \Pi\left[\frac{p^{r(a+1)} - 1}{p^r - 1}\right]; \qquad \zeta_r'(n) = \zeta_r(m);$$

$$Z_r(n) = n^r\Pi\left[1 + \frac{1}{p^r}\right]; \qquad Z_r(2^a m) = 2^{(a-1)r}(2^r + 1)Z_r(m), \alpha > 0;$$

$$Z_r'(2m) = Z_r'(m) = Z_r(m); \qquad Z_r'(2^a m) = 0, \alpha > 1.$$

(v) Closely related to these are the two following: $\alpha_r(n)$, = the sum of the rth powers of all those divisors of n whose conjugates are odd; and the non-factorable $\lambda_r(n)$ defined by

$$\lambda_r(n) = [2(-1)^n + 1]\zeta_r'(n).$$

For the respective conjugates we find, using (18) for Λ_r:

$$A_r(m) = Z_r(m); \qquad A_r(2m) = 2^r Z_r(m);$$

$$A_r(2^a m) = 2^{r(a-2)}(2^{2r} - 1)Z_r(m), \alpha > 1;$$

$$\Lambda_r(m) = -Z_r(m); \qquad \Lambda_r(2m) = 3Z_r(m); \qquad \Lambda_r(4m) = 4Z_r(m);$$

$$\Lambda_r(2^a m) = 0, \alpha > 2.$$

12. As a first illustration of the formulæ we take Eisenstein's theorems.*

* Eisenstein, Crelle, 35 (1847), pp. 134–135.

We shall assume the $T(m)$, all of which, occurring among the forms considered by Liouville,* are proved in Pepin's memoir, and from them deduce the $P(m)$. The forms are

$$x^2 + y^2 + z^2 + 3u^2; \quad x^2 + y^2 + 2z^2 + 2uz + 2u^2; \quad x^2 + y^2 + z^2 + 5u^2;$$

and when m is prime to 3 for the first two, prime to 5 for the third, the several cases of Eisenstein's $T(m)$ may be written for the three forms respectively:

$$T(m) = A\bar{\omega}_1(3, m), B\bar{\omega}_1(3, m), C\omega_1(m, 5);$$

$$A = [2(-1|m) + 1][3(-3|m) - 1],$$

$$B = [2(-1|m) - 1][3(-3|m) + 1],$$

$$C = 5(m|5) + 1.$$

It is convenient to separate A into two cases, A' for $m \equiv 1, 7 \bmod 12$, A'' for $m \equiv 5, 11 \bmod 12$:

$$A' = 2[1 + 2(3|m)], \qquad A'' = 4[2(3|m) - 1].$$

Hence for the first form when $m \equiv 1, 7 \bmod 12$, we have by (22) and § 11 (i),

$$P(m) = 2[\bar{\Omega}_1(3, m) + 2\bar{\Omega}_1'(3, m)] = 2[(3|m) + 2]\bar{\Omega}_1'(3, m);$$

and when $m \equiv 5, 11 \bmod 12$,

$$P(m) = 4[2\bar{\Omega}_1'(3, m) - \bar{\Omega}_1(3, m)] = 4[2 - (3|m)]\bar{\Omega}_1'(3, m).$$

It is readily seen that both are included in either of the single formulæ, the second following at once from the first by § 11 (i),

$$P(m) = (3|m)A\bar{\Omega}_1'(3, m), \qquad P(m) = A\bar{\Omega}_1(3, m),$$

either agreeing with the four cases stated by Eisenstein. Similarly, on separating B in the same way mod 12 into

$$B' = 4[-1 + 2(3|m)], \qquad B'' = 2[1 + 2(3|m)],$$

we find for the second form

$$P(m) = (3|m)B\bar{\Omega}_1'(3, m), \qquad P(m) = B\bar{\Omega}_1(3, m);$$

and for the third, without separation,

$$P(m) = (m|5)C\Omega_1'(m, 5), \qquad P(m) = C\Omega_1(m, 5).$$

Thus in all cases we may pass from T to P by changing ω into Ω, a special case of a general theorem which need not concern us here.

* Journal des Math. (2), vols. 4–11 (1859–1866).

13. The treatment of special characters (§ 10) is exemplified by Liouville's*

$$x^2 + 2y^2 + 4z^2 + 4u^2,$$

for which he states

$$T(m) = 2\omega_1'(m); \qquad T(2^\alpha m) = 2[2^\alpha - (2|m)]\omega_1'(m), \ \alpha > 0.$$

Hence by § 11 (ii), $P(m) = 2\Omega_1'(m)$; and for $\alpha > 0$, as in deriving (18):

$$P(2^\alpha m) = \sum_m [T(d)f(2^\alpha\delta) + T(2d)f(2^{\alpha-1}\delta) + \cdots + T(2^\alpha d)f(\delta)];$$

whence for $\alpha > 1$,

$$P(2m) = \sum_m T(2d)f(\delta), \qquad P(2^\alpha m) = \sum_m [\{- T(2^{\alpha-2}d) + T(2^\alpha d)\}f(\delta)];$$

and as in § 12 on substituting the values of $T(2d)$, $T(2^\alpha d)$ for $\alpha = 2, 3, 4, \cdots$ we find

$$P(2m) = 2[2 - (2|m)]\Omega_1'(m); \qquad P(4m) = 2[3 - (2|m)]\Omega_1'(m);$$
$$P(2^\alpha m) = 3 \cdot 2^{\alpha-1}\Omega_1'(m), \ \alpha > 2,$$

agreeing with the statements of Liouville, loc. cit., § 4.

14. As a last example, let us find the $P(n)$ when (1) is a sum of 4, 6, 8, 10 or 12 squares, from the known factorable $T(n)$, all of the latter having been most simply derived by elliptic functions.†

(i) For four squares we have

$$T(m) = 8\zeta_1(m); \qquad T(2^a m) = 24\zeta_1(m), \ a > 0.$$

* J. des Math. (2), 7 (1862), pp. 62–64. The $P(n)$ for the following papers in the same volume may be proved as in § 13: 145–147; 148–149; 201–204; 205–208.

† See the cited papers of Glaisher and Mordell for theorems and references, to which add the following for 10, 12 squares: Liouville, J. des Math. (2), 11 (1866), pp. 1–8; 6 (1861), pp. 369–377; 233–238. Proofs for the remarkable general $T(n)$ formulæ in these papers of Liouville will appear shortly in the Bulletin of the American Math. Society. Combined with those proofs, the formulæ for $Z_r(n)$, $\Xi_r(n)$, $\Xi_r'(n)$ in § 11 above give at once the proofs for Liouville's general $P(n)$ formulæ, ibid., pp. 373–376. The relation between the 10 and 12 square theorems and the rest of Liouville's $T(n)$ results is simple and striking: the "formules générales" whence Pepin proved the latter are direct paraphrases of trigonometric identities arising from elliptic identities such as $sn^2x = snx \times snx$, when for snx, sn^2x are substituted their Fourier developments and coefficients of q^n equated; the general theorems whence Liouville deduces his 10 and 12 square results come from precisely the same identities when for the elliptic functions and their powers are substituted their power-series expansions, and coefficients of x^n equated. Thus all these apparently diverse results are seen from the standpoint of elliptic functions to be ultimately the same, differing only in algebraic details. The use of the "formules générales" can be avoided entirely, by assigning x the values $\pi/2$, $\pi/3$, $\pi/4$, $\pi/5$, $\pi/6$, $\pi/8$ in the trigonometric identities, the procedure thence onward being obvious from the first sections of Pepin's memoir. As further indicating the connection with elliptic functions of Eisenstein's 10-square and other results, *all* of his assertions concerning 10 squares are proved by Glaisher's formula (i), Q. J. Math., 38 (1906–7), p. 22. The $P(n)$ formulæ for 4 and the $P(m)$ for 6 squares found above agree with those determined arithmetically, cf. Bachmann, Quadr. Formen, pp. 602, 652.

Hence by § 10 and § 11 (iv),

$$P(m) = 8Z_1(m); \qquad P(2m) = 24Z_1(m); \qquad P(4m) = 16Z_1(m);$$
$$P(2^a m) = 0,\ a > 2.$$

(ii) For six squares the known $T(n)$ is

$$T(n) = 4[4\xi_2'(n) - \xi_2(n)];$$

and therefore by § 11 (iii):

$$P(m) = 4[\ 4\Xi_2'(m) - \Xi_2(m)] = 4[\ 4 - (-1\,|\,m)]\Xi_2'(m);$$
$$P(2m) = 4[16\Xi_2'(m) - \Xi_2(m)] = 4[16 - (-1\,|\,m)]\Xi_2'(m);$$
$$P(2^a m) = 15\cdot 2^{2a}\Xi_2'(m),\ a > 1.$$

(iii) For eight squares, $T(n) = -16(-1)^n[2\zeta_3'(n) - \zeta_3(n)]$. Hence

$$T(m) = 16\zeta_3(m); \qquad 7T(2^a m) = 16[2^{3(a+1)} - 15]\zeta_3(m),\ a > 0;$$
$$P(m) = 16Z_3(m); \qquad P(2m) = 112Z_3(m);$$
$$P(2^a m) = 144.2^{3(a-1)}Z_3(m),\ a > 1.$$

(iv) For ten squares the only factorable case is for $n = 2^b m$, $b \geqq 0$, $m \equiv 3 \bmod 4$:

$$T(n) = 4[\xi_4(n) + 16\xi_4'(n)].$$

To find $P(n)$ we have, by the general formulæ,

$$P(m) = \sum_m T(d)f(\delta); \qquad P(2m) = \sum_m T(2d)f(\delta);$$
$$P(2^a m) = \sum_m [T(2^a d) - T(2^{a-2}d)]f(\delta),\ a > 1.$$

The divisors d remaining in these formulæ after reduction will all be of the prescribed form $2^b(4k + 3)$ when and only when m is a prime $\equiv 3 \bmod 4$. Hence *in this case only* we get the following for the numbers of proper representations as a sum of ten squares:

$$P(m) = T(1)f(m) + T(m)f(1) = T(m),$$

the only divisors of m being 1, m. For this value of m, $\xi_4'(m) = -1 + m^4$, $\xi_4(m) = 1 - m^4$; hence $P(m) = 60(m^4 - 1)$. Similarly

$$P(2m) = T(2)f(m) + T(2m)f(1) = T(2m);$$
$$P(2^a m) = T(2^a m) - T(2^{a-2}m),\ a > 1;$$

and hence, after obvious reductions,

$$P(2m) = 1020(m^4 - 1); \qquad P(2^a m) = 255.2^{2(2a-1)}(m^4 - 1),\ a > 1.$$

There are no similar theorems for m not a prime of the form $4k + 3$.

(v) For twelve squares a factorable $T(n)$ exists only when n is even:

$$31T(2^a m) = 24[21 + 2^{5a+1}\cdot 5]\zeta_5(m),\ a > 0.$$

Hence, at once by § 11 (v), we find $P(2m) = 264Z_5(m)$; a result stated without proof by Liouville in a letter to M. Besge (J. des Math. (2), 5 (1860), p. 145). For $a > 1$:

$$P(2^a m) = \sum_m [T(2^a d) - T(2^{a-2}d)]f(\delta).$$

Now $2^a d$, $2^{a-2}d$ will both be even when and only when $a > 2$; and we find, on simplifying,

$$P(2^{a+1}m) = 495\cdot 2^{5a-6}Z_2(m),\ a > 2.$$

15. The extensions of the fundamental formula (11) and § 7 to (3) are obvious and at present of slight interest: it is sufficient to replace $f(n)$ by the factorable $f_s(n)$, which vanishes unless n is the sth power of a simple number, in which case its value is ± 1 according as the multiplicity of n is even or odd.

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Hence by § 10 and § 11 (iv),

$$P(m) = 8Z_1(m); \qquad P(2m) = 24Z_1(m); \qquad P(4m) = 16Z_1(m);$$

$$P(2^a m) = 0, a > 2.$$

(ii) For six squares the known $T(n)$ is

$$T(n) = 4[4\xi_2'(n) - \xi_2(n)];$$

and therefore by § 11 (iii):

$$P(m) = 4[\ 4\Xi_2'(m) - \Xi_2(m)] = 4[\ 4 - (-1 \mid m)]\Xi_2'(m);$$

$$P(2m) = 4[16\Xi_2'(m) - \Xi_2(m)] = 4[16 - (-1 \mid m)]\Xi_2'(m);$$

$$P(2^a m) = 15 \cdot 2^{2a}\Xi_2'(m), a > 1.$$

(iii) For eight squares, $T(n) = -16(-1)^n[2\zeta_3'(n) - \zeta_3(n)]$. Hence

$$T(m) = 16\zeta_3(m); \qquad 7T(2^a m) = 16[2^{3(a+1)} - 15]\zeta_3(m), a > 0;$$

$$P(m) = 16Z_3(m); \qquad P(2m) = 112Z_3(m);$$

$$P(2^a m) = 144.2^{3(a-1)}Z_3(m), a > 1.$$

(iv) For ten squares the only factorable case is for $n = 2^b m$, $b \geqq 0$, $m \equiv 3 \bmod 4$:

$$T(n) = 4[\xi_4(n) + 16\xi_4'(n)].$$

To find $P(n)$ we have, by the general formulæ,

$$P(m) = \sum_m T(d)f(\delta); \qquad P(2m) = \sum_m T(2d)f(\delta);$$

$$P(2^a m) = \sum_m [T(2^a d) - T(2^{a-2}d)]f(\delta), a > 1.$$

The divisors d remaining in these formulæ after reduction will all be of the prescribed form $2^b(4k + 3)$ when and only when m is a prime $\equiv 3 \bmod 4$. Hence *in this case only* we get the following for the numbers of proper representations as a sum of ten squares:

$$P(m) = T(1)f(m) + T(m)f(1) = T(m),$$

the only divisors of m being 1, m. For this value of m, $\xi_4'(m) = -1 + m^4$, $\xi_4(m) = 1 - m^4$; hence $P(m) = 60(m^4 - 1)$. Similarly

$$P(2m) = T(2)f(m) + T(2m)f(1) = T(2m);$$

$$P(2^a m) = T(2^a m) - T(2^{a-2}m), a > 1;$$

and hence, after obvious reductions,

$$P(2m) = 1020(m^4 - 1); \qquad P(2^a m) = 255.2^{2(2a-1)}(m^4 - 1), a > 1.$$

There are no similar theorems for m not a prime of the form $4k + 3$.

(v) For twelve squares a factorable $T(n)$ exists only when n is even:

$$31T(2^a m) = 24[21 + 2^{5a+1}\cdot 5]\zeta_5(m),\ a > 0.$$

Hence, at once by § 11 (v), we find $P(2m) = 264Z_5(m)$; a result stated without proof by Liouville in a letter to M. Besge (J. des Math. (2), 5 (1860), p. 145). For $a > 1$:

$$P(2^a m) = \sum_m [T(2^a d) - T(2^{a-2}d)]f(\delta).$$

Now $2^a d$, $2^{a-2}d$ will both be even when and only when $a > 2$; and we find, on simplifying,

$$P(2^{a+1}m) = 495\cdot 2^{5a-6}Z_2(m),\ a > 2.$$

15. The extensions of the fundamental formula (11) and § 7 to (3) are obvious and at present of slight interest: it is sufficient to replace $f(n)$ by the factorable $f_s(n)$, which vanishes unless n is the sth power of a simple number, in which case its value is ± 1 according as the multiplicity of n is even or odd.

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A PROOF OF JORDAN'S THEOREM ABOUT A SIMPLE CLOSED CURVE.*

By J. W. Alexander.

§ 1. We give below an easy proof of Jordan's theorem that a simple closed curve subdivides the Euclidean plane into two and only two regions.† The argument is based on elementary combinatorial properties of chains, or systems of polygons, and may be generalized at once to any number of dimensions, as will be shown in a subsequent paper.

The necessary properties of chains are somewhat weaker than the properties of polygons which are used as the starting point of most proofs. They are recalled very hastily in Part I, following the general lines exposed by Prof. O. Veblen in a paper "On the Decomposition of an n-Space by a Polyhedron" ‡ which deals with a more inclusive problem and to which we refer for fuller details.

I. Chains and Their Properties.

§ 2. A *chain* will be a sort of generalized polygon consisting of a finite number of non-intersecting *edges* (which may be either line segments or rays), and *vertices* (the end points of the edges), where at each vertex there end an even number of edges. A chain need not be connected.

Suppose we have a chain whose edges are all segments. Then if two vertices, Y and Z, may be joined by a broken line made up of elements of the chain, they may also be joined by a second broken line which has no edge in common with the first. For if we remove from the chain the edges of the first broken line, there will still remain an even number of edges abutting at every vertex except Y and Z where there will now remain an odd number. But, within each connected group of edges and vertices, the total number of times that edges abut on vertices is equal to twice the number of edges and is therefore even. Hence, the vertices Y and Z still belong to the same connected piece and may again be joined by a broken line.

A simple illustration of a chain would be a pair of broken lines con-

* Read before the American Mathematical Society, July, 1916.

† First correctly proved by O. Veblen, Trans. of Amer. Math. Soc., vol. VI, p. 83; 1905. Simplified proofs have been given by L. E. J. Brouwer and others. Cf. Brouwer, Mathematische Annalen, vol. 69, p. 169; 1910.

‡ Trans. of Amer. Math. Soc., vol. XIV, No. 1, pp. 65–72; Jan., 1913.

necting the same two points, Y and Z, and having only a finite number of points in common.

§ 3. A chain, k, like a simple polygon, has two "sides," though the sides are not in general connected regions. We may determine them as follows.*

We complete the lines to which the edges of the chain k belong and thus obtain a system of lines which subdivide one another into a finite number of line segments and rays, $b_1, b_2, \cdots, b_n$, while they subdivide the plane into a finite number of convex regions, $a_1, a_2, \cdots, a_m$. Now, the boundaries of the regions a_i are chains made up of sets of elements b_j and their end points. Out of the symbols for the elements in these sets, we shall form the expressions

$$(1) \qquad a_i = b_{i1} + b_{i2} + b_{i3} + \cdots + b_{ik} \quad (i = 1, 2, \cdots, m),$$

which will be used to designate the boundaries of the various cells a_i. The expressions (1) will be combined by adding corresponding members, collecting terms, and reducing all coefficients modulo 2. In this way, we can obtain new combinations defining new chains whose edges can be read off from the right-hand members.

Now it can be shown without difficulty that any chain such as k composed of elements b_i and their end points can be derived from the elementary chains (1) in two and only two ways,

$$(2) \qquad \Sigma_1 a_i = k$$

and

$$(2') \qquad \Sigma_2 a_i = k,$$

and that the boundary of each region a_i occurs in one and only one of the combinations. Therefore, the points of the plane fall into two classes according as they belong to the interior or boundary of a region occurring in the first combination or of a region occurring in the second. These two classes of points will be called the *sides* of the chain k.

§ 4. We point out one more theorem that will be used later. Suppose we have two chains, k_1 and k_2. Then the set of points which are on given sides both of the chain k_1 and of the chain k_2 may be subdivided into a finite number of convex regions. Therefore, the set is bounded by a chain composed of the sum, modulo 2, of the boundaries of the convex regions. By combining this chain with the chain k_1, we obtain a new chain k_3.

II. **Jordan's Theorem.**

§ 5. In the following discussion, a *region* will be a set of points each of which is interior to a triangle enclosing only points of the set while

* Cf. Veblen, loc. cit., for details.

any two may be joined by an arc made up of points of the set. The first condition is satisfied by the complementary set to any closed set of points in the plane. Moreover, when the second condition is also satisfied, two points, Y and Z, of the region may always be connected within the region by a broken line which may be so chosen as to have only a finite number of points in common with any preassigned finite system of lines, a property which we shall use later on.

To prove this property, we observe that about any point, P, of the arc joining the points Y and Z, we may place a triangle which encloses only points of the region. Therefore, within this triangle, we may find a sub-arc containing the point P, such that any two points of this sub-arc may be joined by a broken line of the required type and such that the point P is not an end point of the sub-arc unless it is an end point of the arc YZ itself. Since the whole arc is covered by these sub-arcs, it may be covered by a finite number of them, by the Heine-Borel theorem. We may therefore construct a broken line connecting the points Y and Z by piecing together a series of broken lines running from one sub-segment to an adjacent one, and so chosen that no two of them have more than a finite number of points in common. The broken line thus obtained may cross itself a finite number of times. When this is the case, we can clearly obtain a broken line without singularities by merely suppressing a certain number of supplementary loops, but for the purposes of this paper there will be no objection in allowing a broken line to have a finite number of singularities.

§ 6. LEMMA. *Let ACB be a simple arc passing through a point C and ending at the points A and B, and let Y and Z be any two points of the plane not on the arc ACB. Then, if the points Y and Z are not separated by either of the sub-arcs AC or CB, either are they separated by the arc ACB itself.*

For the points Y and Z may be connected by a pair of broken lines, a and b, such that the first does not meet the arc AC nor the second the arc CB. Moreover, by § 5, the broken line b may be so chosen as to meet the broken line a in at most a finite number of points, in which case, it may be combined with a to form a chain k.

Now, consider such points of the arc CB as are either on the chain k, (that is, on the broken line a), or on the opposite side of the chain k from the point C. Each of these points may be enclosed within a triangle which neither meets nor encloses a point of the arc AC or of the broken line b, and since the set of all such points is closed, they may all be enclosed within a finite number of these triangles, by the Heine-Borel theorem.

Following § 4, let us add modulo 2 to the chain k the boundaries of the finite set of convex regions made up of points which are both interior

to one of the triangles and on the opposite side of the chain k from the point C. We thus obtain a new chain, k', which still contains the broken line b, as well as a supplementary piece, a', made up of segments which neither meet nor end on the arc ABC. Therefore, by § 2, the points Y and Z may be joined by a broken line within the piece a' and consequently by one which does not meet the arc ACB.

§ 7. THEOREM. *The points of the plane not on a simple arc AB do not form more than one connected region.* For we shall prove that any two such points, Y and Z, may be joined by a broken line which does not meet the arc AB.

About any point, C, of the arc AB, we may place a triangle with respect to which Y and Z are exterior points. Therefore, by remaining within this triangle, we may find a sub-arc of the arc AB which does not separate the points Y and Z, which contains the point C, and which ends at the point C only when C is one of the points A or B. The arc AB may thus be covered by a set of overlapping sub-arcs, and consequently by a finite set of overlapping sub-arcs, such that no one of them separates the points Y and Z. But the end points of this last set subdivide the arc AB into a still smaller finite set of non-overlapping sub-arcs. Therefore, since the arc AB may be built up by piecing together these sub-arcs, it cannot separate the points Y and Z, by § 6.

§ 8. THEOREM. *The points of the plane not on a simple closed curve do not form more than two connected regions.* For we shall prove that, given any three such points, X, Y, and Z, two of them, at least, may always be connected by a broken line which does not meet the curve.

Let A, B, and C be any three distinct points of the curve. Then, by § 7, the points X and Y, Y and Z, and Z and X may be joined by three broken lines, a, b, and c, respectively which do not meet the arcs CAB, ABC, and BCA respectively. Moreover, the broken lines a, b, and c may be so chosen that no two of them have more than a finite number of points in common, so that they may be combined to form a chain, k.

Now, of the three points A, B, and C, two, at least, must be on the same side of the chain k, and we can assume without loss of generality that the points B and C are. It then follows by a direct paraphrase of the reasoning in § 6 that the points Y and Z can be joined by a broken line which does not meet the curve. For we have only to substitute the arc CAB for the arc AC and the broken line bc for the broken line b and repeat the rest of the argument word for word.

§ 9. THEOREM. *The points of the plane not on a simple closed curve form at least two connected regions.*

Choose any two points, A and B, on the curve and denote by AB and

BA respectively the two arcs of the curve bounded by these points. Then any line, l, which separates the points A and B meets the arcs AB and BA in two closed sets of points.

Now, every point of the first set is interior to some interval of the line l which contains no point of the arc BA, hence, by the Heine-Borel theorem, the entire set may be covered by a finite number of such intervals. Moreover, by combining intervals when necessary, we may so arrange that no two of them either touch or overlap. We shall prove that the end points of this last set of intervals, i, are not all within the same region by showing that, however we may connect them in pairs by a system of broken lines, one or more of the broken lines will always meet the curve.

Consider such a system of broken lines, assuming, as we may, that no one of them meets the line l or another broken line of the system in more than a finite number of points. Then the system of lines may be combined with the intervals i to form a chain, k. Moreover, if we add to the chain k the boundary of one side of the line l (that is, the line l itself), we shall obtain a second chain k' made up of the broken lines combined with the segments, s, of the line l complementary to the intervals i. By the definition of the sides of a chain (Relations (2) and (2′), § 4), it is clear that one or other of the chains k and k' separates the points A and B.

Now, if the chain k separates the points A and B, it surely meets the arc BA. But the arc BA cannot meet the intervals i and must therefore meet one of the broken lines of the system. Similarly, if the net k' separates the points A and B, the other arc AB must meet one of the broken lines, since it cannot meet the segments s. Therefore, in either case, the curve meets one of the broken lines, proving that the ends of the intervals i do not all belong to the same region.

§ 10. Corollary. *A point Z not on a simple closed curve may be connected to any arc of the curve AB by a broken line z which, except for one end point, lies wholly within the same region as the point Z.*

For let Y be any point on the opposite side of the curve from Z. Then, by § 6, the points Y and Z may be connected by a broken line which does not meet the other arc BA of the curve and which therefore meets the arc AB. This broken line evidently includes the required broken line z connecting the point Z to a point of the arc AB.

Since the point Z and broken line z may be chosen on either side of the curve, and since the arc AB may be made arbitrarily small, we also have at once the following proposition:

Corollary. In the neighborhood of any point, A, of a simple closed curve, there are points from each side of the curve.

This would also have followed at once from § 9 if we had used an arbitrarily small triangle about the point A instead of the line l.

LINEAR ORDER IN THREE DIMENSIONAL EUCLIDEAN AND DOUBLE ELLIPTIC SPACES.*

By George H. Hallett, Jr.

1. **Introduction.** In this paper I shall deal with Euclidean and double elliptic geometries of three dimensions, the notions of point and order being undefined. A representation of double elliptic geometry of three dimensions is a geometry on the hypersphere

$$x^2 + y^2 + z^2 + w^2 = 1$$

in space of four dimensions. A plane in this geometry is the intersection of this hypersphere with a three-dimensional space through the origin

$$ax + by + cz + dw = 0.$$

The unit sphere in x, y, z-space

$$x^2 + y^2 + z^2 = 1, \qquad w = 0,$$

is a particular plane. A line is the intersection of two distinct planes. In particular all great circles on the above-mentioned unit sphere are lines. Complete treatments of three-dimensional Euclidean and double elliptic geometries in terms of point and order have already been given.†

The problem of this paper may be stated as follows. Two sets of postulates A and B are desired such that

1. Every proposition of Euclidean geometry of three dimensions can be deduced from the postulates of A.

2. Every proposition of double elliptic geometry of three dimensions can be deduced from the postulates of B.

3.‡ No postulates of either set can be derived from the other postulates of that set.

4. All the propositions true for a line in either geometry can be derived from postulates of the set corresponding to the geometry which do not necessitate the existence of a point off the line.

* Presented to the Society, April 27, 1918.

† O. Veblen, "A System of Axioms for Geometry," Transactions of the American Mathematical Society, vol. 5 (1904), pp. 343–384.

J. R. Kline, "Double Elliptic Geometry in Terms of Point and Order Alone," Annals of Mathematics, vol. 18 (1916), pp. 31–44.

‡ I have not completely proved that my Double Elliptic sets satisfy Condition 3. Cf. footnote below concerning $7B$.

5. The sets A and B have in common a set of postulates C.

6. The number of postulates of A and B which do not belong to C is as small as conveniently possible.

7. The postulates of C are natural for both geometries. For example no parts of postulates are vacuously satisfied in either geometry.

8. No postulate of A which does not belong to C is natural for double elliptic geometry, and no postulate of B which does not belong to C is natural for Euclidean geometry.

This problem was suggested by Professor Robert L. Moore, to whom I am likewise indebted for many valuable suggestions in its solution.

I give three pairs of sets of postulates, Σ_1–Σ_3. Use is made of notions of order somewhat different from those used by Veblen and Kline. Veblen's axioms are satisfied by a geometry of ordinary Euclidean space in which three points A, B and C are regarded as having the order ABC if they are all distinct and B lies between A and C on the straight line joining them. Σ_1A, Σ_2A and Σ_3A are satisfied by a Euclidean geometry in which the additional orders AAB, ABB and AAA are true for every pair of points A and B. This notion is due to Kempe.* Kline's axioms are satisfied by a double elliptic geometry in which A, B and C are in the order ABC if B lies between A and C on an arc of less than 180° of one of the great circles which constitute the lines of the geometry.† All my double elliptic sets are satisfied by geometries in which the arc from A to C through B is $\leqq 180°$. Two points which are the end points of an arc of 180° are called opposites. If (a, b, c, d) is a point on the hypersphere

$$x^2 + y^2 + z^2 + w^2 = 1,$$

$(-a, -b, -c, -d)$ is its opposite. Every point of the geometry is between any two such points with my revised notion of order.

When this proposition is established it becomes very powerful in the further development. In Σ_1B, Σ_2B and Σ_3B the orders AAB, ABB and AAA are true for every pair of points A and B.

To show that Σ_1–Σ_3 satisfy conditions 1 and 2 it is sufficient to show that the equivalents of Veblen's axioms for the notions of order used can be deduced from each of the sets Σ_1A, Σ_2A, Σ_3A, and that the equivalents of Kline's axioms can be deduced from each of the sets Σ_1B, Σ_2B, Σ_3B. The categoricity of my sets then follows from the categoricity of Veblen's and Kline's sets.

To show that Condition 3 is satisfied by each set it is sufficient to give for each postulate of the set an independence example, a geometry

* A. B. Kempe, "On the Relation between the Logical Theory of Classes and the Geometrical Theory of Points," Proceedings of the London Mathematical Society, vol. 21 (1890), pp. 147–182.

† This notion was first introduced by Halsted in his Rational Geometry.

in which that postulate is not satisfied but all other postulates of the set are satisfied.*

In this paper I shall make frequent use of Veblen's and Kline's papers. I shall not acknowledge each item separately.

2. **The Set Σ_1.** POSTULATE 1.* *There exist two distinct points.*

DEFINITION.† *B* is *between A* and *C* if *ABC* or *CBA*.

POSTULATE 2.‡ *If A and B are any two distinct points, there exists a point C between A and B and distinct from both.*

POSTULATE 3.§ *If ABA, $B = A$.*

POSTULATE 4.* *If ABC then CBA.*

POSTULATE 5. *If ABC and ACD then BCD.*

POSTULATE 6.† *If ABC and ABD, then either ACD or ADC or $A = B$.*

DEFINITIONS.‡ If *C* is a point in the order *ACB* but not in either of the orders *CBA*, *BAC*, the *line ACB* is the set of all points in some order with *C* and either *A* or *B*. If *A* and *B* are two distinct points of a line *l*, the *segment AB* of *l* is the set of all points of *l* between *A* and *B* other than *A* and *B*, which are called the *end points* of the segment.

POSTULATE 7. SEPARABILITY. *If [S] is an infinite set of distinct points all on the same line, there exists a countable* set of points [K] on that line such that between every two distinct points of [S] there is a point of [K].*

POSTULATE 8. *If [T] is a countably infinite set of distinct segments contained in a segment AB such that (1) every point of AB is in infinitely many segments of [T] and (2) no two distinct points of AB are in infinitely*

* In none of my double-elliptic sets have I succeeded in constructing such an example for Postulate 7*B*.

† This is Veblen's Axiom 1.

‡ In this paper letters denote points unless otherwise stated. The abbreviation "*ABC*" will be used for the expression "points *A*, *B* and *C* are in the order *ABC*."

§ In Veblen's and Kline's treatments this proposition can be proved by means of the two dimensional triangle axiom (Veblen's Axiom 8, Kline's Axiom 7, my Postulate 10), but cannot be proved on the basis of the one-dimensional axioms. Condition 4 is therefore not satisfied in their treatments. In Veblen's treatment the proof involves Axiom 5 (my Theorem 9*A*), in the proof of which Postulate 2 is needed in my treatment.

The expression $B = A$ means that *B* and *A* are the same point. $B \neq A$ means that *B* and *A* are distinct points.

‖ Cf. Veblen's Axiom 4.

¶ The equivalents of Veblen's Axiom 2 and Kline's Axiom 3 are special cases of this postulate.

** Postulates 5 and 6 are used in place of Veblen's Axiom 6, which, if expressed in terms of point and order alone, is somewhat complicated.

†† The definition is given thus in order that the same definition may serve in both geometries and that the mere existence of a point off a line may not imply anything about order on the line. In view of Theorems 3 and 4 and Postulate 4 the condition that *CBA* and *BAC* are false is equivalent in the development from this set of postulates to the condition $A \neq C \neq B$.

‡‡ A *countable* set of points is one which can be brought into one-to-one correspondence with the set of all positive integers or a part of that set. All the points of [*K*] may thus be represented by the sequence $K_1, K_2, K_3, \cdots$.

many segments of $[T]$, *then if* $[T']$ *consists of an infinite number of the segments of* $[T]$, *there exists a point* P *between* A *and* B *such that* (1) *if* $A \neq P \neq B$ *every sub-segment of* AB *containing* P *contains a segment of* $[T']$, *and* (2) *if* $P = A$ (*or* B) *every sub-segment of* AB *determined by* A (*or* B) *and a point of the segment* AB *contains a segment of* $[T']$.

In place of Postulates 7 and 8 Veblen uses the Heine-Borel Theorem (Axiom 11) and Kline uses the Dedekind Cut Postulate (Axiom 10). If it were not for the requirements of Condition 4 either of these propositions might be used here with slight modifications instead of Postulate 8. Examples 7 would serve as independence examples for either. But in his paper entitled Definition in Terms of Order Alone in the Linear Continuum and in Well-ordered Sets (Transactions of the American Mathematical Society, vol. 6, 1905, pp. 165–171) Veblen gives an example for his uniformity postulate with the aid of which it may be seen that separability would not follow from the linear postulates of the set obtained from Σ_1 by replacing Postulate 8 by the Heine-Borel or Dedekind Cut Postulate. Moreover separability could not be assumed here in addition to the Dedekind Cut Postulate without destroying the independence of the set. For Professor Robert L. Moore has shown that separability follows from the Dedekind Cut Postulate and the order postulates with the aid of the two-dimensional triangle postulate (my Postulate 10). This was done in his paper On a Set of Postulates which Suffice to Define a Number Plane, Transactions of the American Mathematical Society, vol. 16 (1915), pp. 27–32. Separability could not be assumed in addition to the Heine-Borel Theorem for a similar reason. The difficulty is obviated by assuming Postulate 8 in addition to separability and proving the Dedekind Cut Postulate and the Heine-Borel Theorem with the aid of these two linear postulates. This solution of the problem is due to Professor Moore.

Postulate 9. *No line contains all points.*

Postulate 10. *If* ABC *and* BDE *are two lines and* E *is not on the line* ABC, *there exists a point* F *between* A *and* E *and on a line containing* C *and* D.

Definitions. If $A \neq B$ and C is not on a line containing A and B, the *triangle* ABC consists of all points between two of the points A, B, C. The *plane* ABC consists of all points of all lines containing two distinct points of the triangle ABC.

Postulate 11. *No plane contains all points.*

Definitions. If A, B and C determine a plane and D is not in that plane, the *tetrahedron* $ABCD$ consists of all points which are between two points of a triangle determined by three of the points A, B, C, D.

The *space* $ABCD$ consists of all points of all lines which contain two distinct points of the tetrahedron $ABCD$.

POSTULATE 12. *If four points determine a space, there is no point which does not belong to that space.*

POSTULATE 13A.* *If A, B, C and D are four distinct points, then there exists a point E in the order ABE and not between C and D.*

POSTULATE 14A. *If a is any line of any plane α and there is a point of α not on a, then there is some point of α not on a through which there is not more than one line of α which contains no point of a.*

POSTULATE 13B. *If A, B and C are any three points and there exist points X and Y in the orders ABX and ACY,† then there exists a point D in the orders ABD and ACD.*

INDEPENDENCE EXAMPLES.

EXAMPLE 1.‡ No points. No order.

EXAMPLE 2. Two points. No order.

EXAMPLE 3. Three distinct points. All possible orders.

EXAMPLE 4A. Euclidean geometry§ with the following modification. There is a unique line l through a unique point A. BCA is never true if B and C lie on l to the left of A and A, B and C are all distinct. All other usual orders are true.

EXAMPLE 4B. Double elliptic geometry with the modification that on a certain line there are no clockwise orders except those involving opposites or two identical points.

EXAMPLES 5A AND 5B. Euclidean and double elliptic geometries with the modification that AAA is never true.

EXAMPLES 6A AND 6B. Euclidean and double elliptic geometries with the convention that ABC implies that A, B and C are all distinct.

EXAMPLE 7A. $[G]$ is a well-ordered set of symbols in one-to-one correspondence with the set of all points on a line. All rational functions of the symbols of $[G]$ are called numbers. Two such numbers are equal if one can be reduced to the other by ordinary algebraic operations. To compare two polynomials in the symbols of $[G]$ proceed as follows. Let $G_1, G_2, \cdots G_n$ be the symbols of $[G]$ involved so arranged that G_i follows

* This modification of Veblen's Axiom 5 is useful not merely in satisfying Condition 4, but also in proving another four-point proposition (Theorem 11) from Postulates 5 and 6.

† This condition is imposed in order that Postulate 13B may not imply Theorem 3 which is easily proved from postulates of the common basis.

‡ Independence examples are numbered to correspond to the postulates whose independence they prove. When, as in this case, the same example will serve in both systems, no letter is used. Otherwise the example in $\Sigma_1 A$ is marked A and the example in $\Sigma_1 B$ is marked B.

§ "Euclidean geometry" or "double elliptic geometry" will mean in this paper the geometry given by the full set of postulates under consideration.

G_{i+1} for $i = 1, 2, \cdots n-1$. Regard $G_2, \cdots G_n$ as positive constants and allow G_1 to increase indefinitely. If as G_1 increases one polynomial becomes and remains greater than the other irrespective of the values assigned to $G_2, \cdots G_n$, it is greater. Otherwise regard $G_1, G_3, G_4, \cdots G_n$ as positive constants and allow G_2 to increase indefinitely. If one polynomial thus becomes and remains greater than the other, it is greater. Otherwise continue this process as many times as may be necessary to reach a decision. A decision will always be reached unless the two polynomials are equal. A positive polynomial is one greater than zero. To compare two fractional functions of the symbols of $[G]$, reduce to a common positive denominator and compare numerators. The number system thus ordered satisfies Postulates 1–6. It is not separable and the Dedekind Cut Postulate fails. Postulate 8 is vacuously satisfied.

On this number system a three dimensional geometry similar to Euclidean geometry may be built as follows. A point is a set of three numbers (a, b, c). A point (a, b, c) satisfies an equation

$$f(x, y, z) = 0$$

if the equation is satisfied when the substitutions $x = a$, $y = b$, $z = c$ are made. The set of all points satisfying a particular linear equation

$$c_1x + c_2y + c_3z + c_4 = 0$$

is called a plane. A line is the set of all points common to two distinct planes. Points $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$ and $C(x_3, y_3, z_3)$ have the order ABC if $A = B$ or $B = C$, or one of the sets of inequalities

$$\begin{array}{ll} x_1 > x_2 > x_3, & x_1 < x_2 < x_3, \\ y_1 > y_2 > y_3, & y_1 < y_2 < y_3, \\ z_1 > z_2 > z_3, & z_1 < z_2 < z_3, \end{array}$$

is true and A, B and C are on the same line.

Examples 8A and 8B. Rational Euclidean and double elliptic geometries.

Example 9A. One-dimensional Euclidean geometry. Geometry on a line.

Example 9B. One-dimensional double elliptic geometry. Geometry on a circle.

Example 10A. Euclidean geometry with the following modification. There are two distinct peculiar points, A and B; A is substituted for B and B for A in any order involving at least one point off the line AB.

Example 10B. Two planes divide the remainder of double elliptic

space into two pairs of precisely similar divisions. The pairs are not similar unless the planes are orthogonal. All points of one pair are retained, all points of the other discarded. All points of one bounding plane are retained, all points of the other which are not points of the first are discarded. Order among the remaining points is the same as in the full double elliptic system.

EXAMPLE 11*A*. Two-dimensional Euclidean geometry. Geometry on a plane.

EXAMPLE 11*B*. Two-dimensional double elliptic geometry. Geometry on a sphere.

EXAMPLES 12*A* AND 12*B*. Euclidean and double elliptic geometries of four dimensions.

EXAMPLE 13*A*. Double elliptic geometry.

EXAMPLE 14*A*.* Bolyai-Lobatchewskian geometry.

EXAMPLE 13*B*. Euclidean geometry.

THEOREMS DERIVED FROM THE COMMON BASIS.

THEOREM 1. *If* $A \neq B$, ABB.

Proof. There exists C in the order ACB such that $A \neq C$ (p.† 2 and p. 4). Since ACB and ACB and $A \neq C$, ABB (p. 6).

THEOREM 2. *If A is any point*, AAA.

Proof. There exists B distinct from A (p. 1). BAA (th. 1). $\therefore AAA$ (p. 5).

THEOREM 3. If A and B are any two points, ABB and AAB.

Proof. ABB and BAA (ths. 1 and 2). $\therefore AAB$ (p. 4).

THEOREM 4. *If ABC and ACB*, $B = C$.

Proof. BCB (p. 5). $\therefore B = C$ (p. 3).

COROLLARY.‡ *If ABC, BCA is not true unless $B = C$.*

THEOREM 5.§ *There exist three distinct points in the order ABC.*

Proof. There exist two distinct points, A and C (p. 1). There exists B distinct from both in the order ABC (p. 2 and p. 4).

THEOREM 6. *If ABC and ACD, ABD.*

Proof. BCD (p. 5). DCA and DCB (p. 4). $\therefore DAB$ or DBA or $D = C$ (p. 6). If $D = C$, ABC means ABD. If DAB, DCA gives CAB (p. 5). Since CBA (p. 4), $A = B$ (th. 4). $\therefore ABD$ (th. 3). If DBA, ABD (p. 4). Thus in any case, ABD.

* In view of this example the common basis here given will serve as a common basis for Euclidean, Bolyai-Lobatchewskian and double elliptic geometries, and the common basis together with Postulate 13*A* will serve as a common basis for Euclidean and Bolyai-Lobatchewskian geometries.

† In proof references, p. = Postulate, th. = Theorem.

‡ Cf. Veblen's Axiom 3.

§ This is the equivalent of Kline's Axiom 2.

THEOREM 7.* *If ABC and ACD and $B \neq C$, $B \neq D$.*

Proof. If $D = B$, ACB. But ABC. $\therefore B = C$ (th. 4), contrary to hypothesis.

THEOREM 8. *If ABC and ABD, BCD or BDC or $A = B$.*

Proof. ACD or ADC or $A = B$ (p. 6). If ACD, ABC gives BCD (p. 5). If ADC, ABD gives BDC (p. 5).

Theorem 4 may be substituted in Σ_1 for Postulate 3. If this is done the same independence examples will serve and the development proceeds as follows.

THEOREMS 1–3 as before.

POSTULATE 3′. *If ABC and ACB, $B = C$.*

THEOREM 4′. *If ABA, $B = A$.*

Proof. If $A \neq B$ then AAB (th. 1 and p. 4). But ABA. $\therefore B = A$ (p. 3′).

The development then continues as before. Postulates 3 and 3′ are thus seen to be equivalent in the presence of Postulates 1, 2, 4–12, 13A and 14A or Postulates 1, 2, 4–12 and 13B, or in fact Postulates 2, 4, 5 and 6, the only other postulates used in proving Postulate 3′ from Postulate 3 and Postulate 3 from Postulate 3′.

EUCLIDEAN GEOMETRY FROM $\Sigma_1 A$.

THEOREM 9.† *If A and B are any two points, there exists a point C distinct from B in the order ABC.*

Proof. Case 1. $A = B$. There exists C distinct from B (p. 1). ABC (th. 3).

Case 2. $A \neq B$. There exist points D and E such that $ADB[A \neq D \neq B]$ and $AED[A \neq E \neq D]$ (p. 2 and p. 4). $B \neq E$, for otherwise $B = D$ follows from ADB and ABD (th. 4). $\therefore A, B, D$ and E are all distinct. There exists therefore a point C in the order ADC and not between E and B (p. 13A). AED and ADC. $\therefore AEC$ (th. 6). ADC and ADB and $A \neq D$. $\therefore ACB$ or ABC (p. 6). If ACB, AEC gives ECB (p. 5), contrary to hypothesis. $\therefore ABC$. Since C is not between E and B, $C \neq B$ (th. 3).

THEOREM 10. *If ABC and ADC, ABD or ADB.*

Proof. There exists E distinct from A in the order CAE (th. 9). CBA and CDA (p. 4). $\therefore EAB$ and EAD (p. 5 and p. 4). $\therefore ABD$ or ADB (th. 8).

THEOREM 11. *If ABC and BCD then either ABD or $B = C$.*

Proof Case 1. $A = B$. ABD (th. 3).

Case 2. $A = C$. CBC. $B = C$ (p. 3).

* Cf. Theorems 6 and 7 with Kline's Axiom 5.

† Cf. Theorem 9, Case 2 with Veblen's Axiom 5.

Case 3. $A = D$. ABC and ACB (p. 4). $B = C$ (th. 4).

Case 4. $B = D$. BCB. $B = C$ (p. 3).

Case 5. $C = D$. ABD.

Case 6. A, B, C and D all distinct. There exists E in the order ACE and not between B and D. ABC and ACE. $\therefore BCE$ (p. 5) and ABE (th. 6). BCD. $\therefore BED$ or BDE (p. 6). But E is not between B and D by hypothesis. $\therefore BDE$. EDB and EBA (p. 4). $\therefore ABD$ (p. 5 and p. 4*).

THEOREM 12.† *If A and B are any two distinct points there is just one line ACB and a necessary and sufficient condition that a point X be on this line is that it be in one of the orders XAB, AXB, ABX.*

Proof. Let C be any point such that ACB, but not CBA or BAC. Then ACB is a line. C is distinct from A and B (th. 3). Let X be any point on the line ACB.

Case 1. XAC. ACB. $\therefore XAB$ (th. 11).

Case 2. AXC. ACB. $\therefore AXB$ (th. 6).

Case 3. ACX. ACB. $\therefore CXB$ or CBX (th. 8). See Cases 5 and 6.

Case 4. XCB. ACB. $\therefore XAC$ or AXC (th. 8). See Cases 1 and 2.

Case 5. CXB. ACB. $\therefore AXB$ (th. 6).

Case 6. CBX. ACB. $\therefore ABX$ (th. 11).

The necessity of the condition is thus established.

Now let X be any point in one of the orders XAB, AXB, ABX, and C as before any point such that ACB, but not CBA or BAC.

Case 1. XAB. ACB. $\therefore XAC$ (p. 5).

Case 2. AXB. ACB. $\therefore AXC$ or ACX (th. 10).

Case 3. ABX. ACB. $\therefore CBX$ (p. 5).

The sufficiency of the condition is thus also established. As the condition stated in the theorem does not involve C and has been proved equivalent to the definition of a line, it follows that there is not more than one line ACB. As there is always a point C (p. 2 and th. 4) and a point in some order with A and C (th. 3), one line always exists.

THEOREM 13.‡ *If C and D are any two distinct points of the line AB, A is a point of the line CD.*

Proof. C is in one of the orders $(\alpha)CAB$, $(\beta)ACB$, $(\gamma)ABC$ (th. 12). D is in one of the orders $(a)DAB$, $(b)ADB$, $(c)ABD$.

Case αa. BAC and BAD. $\therefore ACD$ or CDA (th. 8).

* Hereafter Postulate 4 will be used without reference.

† This theorem establishes the equivalence of Veblen's and my definitions for a line. In view of this equivalence the equivalents of Veblen's Axioms 7, 9 and 10 follow from my Postulates 9, 11 and 12.

‡ This theorem is the equivalent of Veblen's Axiom 6 in view of Theorem 12. The proof of this theorem illustrates the advantage of the Kempe notion of order. If Veblen's order were used, 25 cases would have to be considered instead of 9.

Case αb. BAC and BDA. $\therefore CAD$ (p. 5).
Case αc. CAB and ABD. $\therefore CAD$ (th. 11).
Case βa. BCA and BAD. $\therefore CAD$ (p. 5).
Case βb. ACB and ADB. $\therefore ACD$ or CDA (th. 10).
Case βc. ACB and ABD. $\therefore ACD$ (th. 6).
Case γa. CBA and BAD. $\therefore CAD$ (th. 11).
Case γb. ABC and ADB. $\therefore CDA$ (th. 6).
Case γc. ABC and ABD. $\therefore ACD$ or CDA (p. 6).

Thus in any case A is on the line CD (th. 12).

THEOREM 14. *If C and D are distinct points of the line AB, the line CD is the line AB.*

Proof. Suppose $C \neq A$. Then the lines AC and AB are identical. For let X be any point of AB. If $X = C$, AXC (th. 1) and X is on AC. If $X \neq C$, A is on CX (th. 13). $\therefore AXC$, XAC or XCA. $\therefore X$ is on AC. Similarly, since B is a point of AC (th. 13), every point of AC is a point of AB. The lines AC and CD are similarly identical. Therefore the lines AB and CD are identical.

If $C = A$, $D \neq A$. By reasoning similar to the above,

$$AB = AD = CD.$$

COROLLARY. *Two distinct lines cannot have more than one point in common.*

For the sake of brevity I shall assume in the remainder of the development certain properties of linear order which follow easily from those already established.

THEOREM 15.* *If $ABC[A \neq B \neq C]$ and $BDE[B \neq D \neq E]$ and E is not on the line, AB then $C \neq D$ and there exists a point F on the line CD such that $EFA[E \neq F \neq A]$.*

Proof. Line ABC = line AC (th. 12). B is on AC and distinct from A. $\therefore ABC = AC = AB$ (th. 14).

$C \neq D$. For if $C = D$, AB and EB have in common two distinct points B and C, and are therefore identical (th. 14). But E is on EB and not on AB by hypothesis.

There exists a point F on the line CD between A and E (p. 10).

$A \neq F$. For if $A = F$, AB and CD have in common two distinct points A and C and are therefore identical. The lines AB and BE have only B in common. Since D is on BE and is distinct from B, D is not on AB. But D is on CD. $\therefore AB \neq CD$.

$E \neq F$. For if $E = F$, CD and BE have in common the two distinct points D and E. $\therefore CD = BE$. BE has only B in common with AB. But CD has C in common with AB. $\therefore CD \neq BE$.

* Cf. Veblen's Axiom 8.

THEOREM 16. DEDEKIND CUT POSTULATE. *If all the points of a segment AB are divided into two sets $[S_1]$ and $[S_2]$ such that no point of either is between two points of the other, there exists a point H between every point of $[S_1]$ and every point of $[S_2]$.*

Proof. There exists a countable set of points $[K]$ on the segment AB such that between every two distinct points of AB there is a point of $[K]$ (p. 7). The set of all segments determined by pairs of points of $[K]$ is countable. Let this set be A_1B_1, A_2B_2, A_3B_3, $\cdots$. Let A, C_{i1}, C_{i2}, $\cdots$ $C_{im_i}B$ be for every integer i the set of points A, B, A_1, B_1, A_2, B_2, $\cdots$ A_i, B_i arranged in the order in which they occur on the line AB, each point being counted only once. A countable set of segments $[T]$ is determined as follows. $T_1 = AC_{12}$, $T_2 = AC_{11}$, $T_3 = C_{11}B$, $T_4 = C_{11}C_{12}$, $T_5 = T_{l_1}{}^* = C_{12}B$, $T_{l_1+1} = AC_{22}$, $T_{l_1+2} = AC_{21}$, $T_{l_1+3} = C_{21}C_{23}$, $T_{l_1+4} = C_{21}C_{22}$, $\cdots$ $T_{l_1+l_2} = C_{2m_2}B$, $T_{l_1+l_2+1} = AC_{32}$, $T_{l_1+l_2+2} = AC_{31}$, $T_{l_1+l_2+3} = C_{31}C_{33}$, etc. One of the segments T_1, T_2, $\cdots$ T_5 has one end point in $[S_1]$, the other in $[S_2]$. Call the T of lowest subscript having this property T_1'. Similarly one of the segments $T_{l_1+\cdots+l_i+1}$, $T_{l_1+\cdots+l_i+2}$, $\cdots$ $T_{l_i+\cdots+l_i+l_{i+1}}$ has one end point in $[S_1]$ and the other in $[S_2]$, and the segment of lowest subscript in this set which has this property is called T_{i+1}'.

Every point of AB is contained in infinitely many of the segments of $[T]$, for every point of AB is contained in one of the segments $T_{l_1+\cdots+l_i+1}$, $T_{l_1+\cdots+l_i+2}$, $\cdots$ $T_{l_1+\cdots+l_i+l+l_{i+1}}$ for every integer i.

No two distinct points P and Q of AB are contained in infinitely many of the segments of $[T]$. For there exist distinct points K_1, K_2 and K_3 of $[K]$ in the order $PK_1K_2K_3Q$. Either the segment K_1K_2 or one included in it is a segment T_j of $[T]$. No segment of $[T]$ following T_j in the sequence T_1, T_2, T_3, $\cdots$ contains both P and Q.

There exists a point H between A and B such that (1) if $A \neq H \neq B$, every segment of AB containing H contains a segment of $[T'] = T_1'$, T_2', $\cdots$, and (2) if $H = A$ (or B) every segment of AB having H as one end point contains a segment of $[T']$ (p. 8). Either AS_1S_2B or AS_2S_1B is true for all points of $[S_1]$ and all points of $[S_2]$. We may assume without loss of generality that AS_1S_2B is always true. Suppose $A \neq H \neq B$. Then H belongs to either $[S_1]$ or $[S_2]$. If it belongs to $[S_1]$ there is no point of $[S_1]$ distinct from H in the order AHS_1B, for the segment AS_1 in that case would contain H but no segment of $[T']$. Similarly if H is a point of $[S_2]$, there is no point of $[S_2]$ distinct from H in the order AS_2HB. Therefore H is between every point of $[S_1]$ and every point of $[S_2]$ unless $H = A$ or B. But $H \neq A$, for otherwise, if S_1 is any point of $[S_1]$, the segment AS_1 contains no segment of $[T']$. Similarly $H \neq B$.

* Here $l_1 = 5$.

It has been shown by Veblen that the Heine-Borel Theorem, Veblen's Axiom 11, follows from the Dedekind Cut Postulate and certain other properties of linear order which have been established.*

The equivalents of all of Veblen's axioms have now been proved on the basis of $\Sigma_1 A$. $\Sigma_1 A$ is therefore categorical.

Double elliptic geometry from $\Sigma_1 B$.

Theorem 9. *Every point is between two points distinct from it.*

Proof. Let A be any point. There exists D distinct from A (p. 1), and B such that $ABD[A \neq B \neq D]$ (p. 2). There exists C such that BAC and BDC (th. 1 and p. 13B). If $C = A$, ADB. But ABD. $\therefore B = D$ (th. 4), contrary to hypothesis. $\therefore A$ is between B and C and distinct from both of them.

Definition. A' is said to be an opposite of A if there is no point B distinct from A' in the order $AA'B$.

Theorem 10.† *Every point has an opposite.*

Proof. Let A be any point. A is between two points C and D distinct from it (th. 9). There exists a point A' in the order ACA' and ADA' (p. 13B). Suppose B is a point distinct from A' in the order $AA'B$. Since ACA' and ADA', $CA'B$ and $DA'B$ (p. 5). $\therefore A'CD$ or $A'DC$ (th. 8). Since $A'CA$ and $A'DA$, CDA or DCA (p. 5). Since CAD, $D = A$ or $C = A$ (th. 4), contrary to hypothesis. Therefore there is no point B distinct from A' in the order $AA'B$, and A' is an opposite of A.

Corollary. *If $ABC[A \neq B \neq C]$, BAD and BCD, then D is an opposite of B.*

Theorem 11. *No point has more than one opposite.*

Proof. Suppose A' and A'' are opposites of A. There exists a point X in the orders $AA'X$ and $AA''X$ (p. 13B). By the definition of an opposite, $X = A' = A''$.

Corollary. *If A' is the opposite of A, $A' \neq A$.*

Theorem 12. *If A' is the opposite of A, every point is between A and A'.*

Proof. Let P be any point. If $P = A$ or A', APA' (th. 3). Suppose $A \neq P \neq A'$. There exists a point Q in the orders PAQ and $PA'Q$ (p. 13B). $Q \neq A$, for otherwise $AA'P$ and $P = A'$ (by definition), contrary to hypothesis. There exists a point A'' in the orders APA'' and AQA'' (p. 13B), and A'' is an opposite of A (th. 10, cor.). $\therefore A'' = A'$ (th. 11). $\therefore APA'$.

Theorem 13. *If A' is the opposite of A, A is the opposite of A'.*

Proof. If A is not the opposite of A', there exists B such that $A'AB$

* Cf. O. Veblen, "The Heine-Borel Theorem," Bulletin of the American Mathematical Society, vol. 10 (1903–04), pp. 436–439.

† Cf. Kline's Axiom 1.

and $B \neq A$. But $A'BA$ (th. 12). $\therefore A = B$ (th. 4), contrary to hypothesis.

THEOREM 14.* *If ABC, then $A'CB$, $B'A'C$ and $A'B'C'$.*

Proof. ACA' (th. 12). $\therefore BCA'$ (p. 5). Similarly $B'A'C$ and $A'B'C'$.

THEOREM 15. *If ABC and ADC, then ABD or ADB or $C = A'$.*

Proof. $A'CB$ and $A'CD$ (th. 14). $\therefore A'DB$ or $A'BD$ or $C = A'$ (p. 6). $\therefore ABD$ or ADB (th. 14) or $C = A'$.

THEOREM 16. *If ABC and BCD, then ABD or $AB'D$ or $B = C$.*

Proof. BCA' (th. 14) and BCD. $\therefore A'DB$ or $DA'B$ or $B = C$ (p. 6). $\therefore ABD$ or $AB'D$ (th. 14) or $B = C$.

THEOREM 17.† *If A and B are any two distinct non-opposite points, there is just one line ACB and a necessary and sufficient condition that a point X be on this line is that either XAB, AXB, ABX or $AX'B$.*

Proof. Let C be any point distinct from A and from B such that ACB. By Theorem 4 neither CBA nor BAC. Then ACB is a line. Let X be any point on the line ACB.

Case 1. XAC. ACB. $\therefore XAB$ or $XA'B$ (th. 16). If $XA'B$, $AX'B$ (th. 14).

Cases 2–5 are the same as Cases 2–5 respectively in the proof of Theorem 12A.

Case 6. CBX. ACB. $\therefore ABX$ or $AB'X$ (th. 16). If $AB'X$, $AX'B$ (th. 14).

The necessity of the condition is thus established.

Now let X be any point such that either XAB, AXB, ABX or $AX'B$ and let C be selected as above.

Cases 1–3 are the same as Cases 1–3 in the proof of Theorem 12A, except that Theorem 15B replaces Theorem 10A in Case 2.

Case 4. $AX'B$. ACB. $AX'C$ or ACX' (th. 15). If $AX'C$, $X'CB$ (p. 5) and CBX (th. 14). If ACX', XAC (th. 14).

The sufficiency of the condition is thus established also. It follows (cf. proof of Theorem 12A) that any two distinct non-opposite points determine just one line.

THEOREM 18.‡ *If C and D are any two distinct non-opposite points of the line AB§, A is a point of the line CD.*

* Cf. Kline's Axiom 4.

† This theorem establishes the equivalence of Kline's and my definitions for a line. In view of this the equivalents of Kline's Axioms 6, 8 and 9 follow directly from Postulates 9, 11 and 12 respectively.

‡ Here again the Kempe notion of order is of great advantage. Kline had 64 cases to consider instead of 16. If he had adopted the notion that every point is between any two opposites, but not the Kempe notion, he would have had 36 cases.

§ "The line AB" implies that A and B are distinct and not opposites, so that they determine a single line.

Proof. One of the orders $(\alpha)CAB$, $(\beta)ACB$, $(\gamma)ABC$, $(\delta)AC'B$ is true, and one of the orders (a) DAB, (b) ADB, (c) ABD, (d) $AD'B$ is true.

Cases αa–c, βa–c, γa–c are similar to the same cases in the proof of Theorem 13*A*.

Case αd. CAB and $AD'B$. $\therefore CAD'$ (p. 5). ACD (th. 14).

Case βd. ACB and $AD'B$. $\therefore ACD'$ or $AD'C$ (th. 15). CAD or $CA'D$ (th. 14).

Case γd. ABC and $AD'B$. $\therefore AD'C$ (th. 6). $CA'D$ (th. 14).

Case δa. $AC'B$ and DAB. $\therefore DAC'$ (p. 5). CDA (th. 14).

Case δb. $AC'B$ and ADB. $\therefore AC'D$ or ADC' (th. 15). $CA'D$ or CAD (th. 14).

Case δc. $AC'B$ and ABD. $\therefore AC'D$ (th. 6). $CA'D$ (th. 14).

Case δd. $AC'B$ and $AD'B$. $\therefore AC'D'$ or $AD'C'$ (th. 15). CDA or ACD (th. 14).

THEOREM 19. *If C and D are distinct non-opposite points of the line AB, the line CD is the line AB.*

The proof is similar to the proof of Theorem 14*A*.

THEOREM 20. *Two distinct lines have at most a point and its opposite in common.*

Proof. If the lines are lines determined by pairs of non-opposite points, the theorem follows immediately from Theorem 19. Consider the line $ABA'[A \neq B \neq A']$. Any point X on this line is in one of the orders XAB, AXB, ABX, $XA'B$, $A'XB$, $A'BX$. $\therefore XAB$, AXB, ABX or $AX'B$ (th. 14). $\therefore$ line ABA' = line AB. Therefore all cases reduce to the case first considered.

THEOREM 21.* *If $ABC[A \neq B \neq C \neq A']$ and $BDE[B \neq D \neq E \neq B']$ and E is not on the line AB, there exists a point F such that $CDF[C \neq D \neq F \neq C']$ and $EFA[E \neq F \neq A \neq E']$.*

Proof. By reasoning similar to the proof of Theorem 15*A*, there exists F on the line CD $[C \neq D \neq C']$ and such that $EFA[E \neq F \neq A]$. $A \neq E'$ since E is not on the line AB.

$F \neq C$, for otherwise $AB = AE$ (th. 20), contrary to the hypothesis that E is not on AB. Similarly $F \neq C'$.

$F \neq D$, for otherwise $AE = BE$ (th. 20). $AB \neq BE$ since E is not on AB. Therefore AB and BE have only B and B' in common (th. 20). But if $AE = BE$, they must have A in common also. Similarly $F \neq D'$.

Since F is on CD, one of the orders FCD, CFD, CDF, $CF'D$ is true.

Suppose FCD. AFE. D is not on AE, by the reasoning used to prove that $F \neq D$. Therefore there exists X between E and D and on the line AC (p. 10). Since the lines ED and AC have only the points

* Cf. Kline's Axiom 7.

B and B' in common (th. 20), $X = B$ or B'. $\therefore EBD$ or $EB'D$. But EDB and DEB' (th. 14). $\therefore B = D$ or $E = B'$ (th. 4), contrary to hypothesis.

Suppose CFD. Then FDC' (th. 14). AFE. Since $AC' = AB$ and E is not on AB, C' is not on AE (th. 18). Therefore there exists X on ED and between A and C' (p. 10). Since AC' and ED have only B and B' in common, $X = B$ or B'. $\therefore ABC'$ or $AB'C'$. $\therefore CAB$ or ACB (th. 14). But CBA. $\therefore A = B$ or $B = C$ (th. 4), contrary to hypothesis.

Suppose $CF'D$. Then $FC'D$ (th. 14). AFE. D is not on AE. Therefore there exists X between D and E and on the line $AC' = AB$ (p. 10). Since DE and AB have only B and B' in common, $X = B$ or B'. $\therefore DBE$ or $DB'E$. But EDB and DEB' (th. 14). $\therefore B = D$ or $E = B'$, contrary to hypothesis. Therefore CDF.

The proof of the equivalent of Kline's Axiom 10 is similar to the proof of Theorem 16A.

The equivalents of all of Kline's axioms have now been established on the basis of $\Sigma_1 B$, which is therefore categorical.

3. **The Set Σ_2.** Σ_2 is a set in which Postulate 4 of Σ_1 is replaced by a third four-point postulate. Postulates 1–3, 5, 7–12, 13A, 14A, 13B are the same as the corresponding postulates in Σ_1 except that the first definition is modified as follows.

DEFINITION. B is *between* A and C if ABC.

POSTULATE 4. *If ABC and BDC, CDA.*

POSTULATE 6. *If ABC and DBC, ADC or DAC or $B = C$.*

INDEPENDENCE EXAMPLES.

All examples are those given under Σ_1 except the following.

EXAMPLES 4A AND 4B. Euclidean and double elliptic geometries with the modification that ABB is never true.

DEVELOPMENT.

THEOREM 1. *If $A \neq B$, AAB.*

Proof. There exists C distinct from B in the order ACB (p. 2). Since ACB and ACB and $C \neq B$, AAB (p. 6).

THEOREM 2.* *If ABC, CBA.*

Proof. If $C = A$, $B = A$ (p. 3). CBA means ABC, which is true by hypothesis.

If $C \neq A$, AAC (th. 1). Also ABC. $\therefore CBA$ (p. 4).

THEOREM 3.† *If ABC and ABD, ACD or ADC or $A = B$.*

* Postulate $\Sigma_1 4$.

† Postulate $\Sigma_1 6$.

Proof. CBA and DBA (th. 2). $\therefore CDA$ or DCA or $A = B$ (p. 6). $\therefore ACD$ or ADC or $A = B$ (th. 2).

Σ_1 has now been proved on the basis of Σ_2, which is therefore categorical.

As before Postulate 3 may be replaced by Postulate 3′. Postulate 3 may be proved from Postulate 3′, with the help of Postulates 2 and 6, as follows.

If ABA, $B = A$.

Proof. If $A \neq B$, AAB (th. 1). ABA by hypothesis. $\therefore B = A$ (p. 3′), contrary to supposition.

4. **The Set Σ_3.** Although Postulate $\Sigma_1 10$ has been so phrased as to apply only in space of two or more dimensions, it applies in one dimension also if stated simply as follows.

If ABC and BDE, there exists a point F in the orders CDF and AFE.

This is due of course to the revised notion of order used. The same is true of Veblen's Theorem 13, which is used as a postulate in this set.

Postulates 1–3, 6–10 and 11A are the same as Postulates 1–3, 7–9, 11, 12 and 13A respectively of Σ_2, except that the definition of a plane is modified.

Postulate 4. *If ABC and DBC, ADC or DAC or $B = C$.*

Postulate 5. *If ABC and ADE, there exists a point F in the orders BFE and CFD.*

Definition. If $A \neq B$, and C is not on a line containing A and B, the *plane* ABC consists of all points of all lines containing two distinct points in some order with two of the points A, B, C.

Postulate 12A.* *If l is a line and C is any point, there is not more than one line in a plane containing l and C and containing C but no point of l.*

Postulate 11B. *If A, B and C are distinct points, there exists a point D in the orders ABD and ACD.*

Independence examples.

Examples 1–3, 6, 7A, 8–10, 11A, 12A, 11B are the same as the examples for the corresponding postulates of Σ_1.

Example 4A. Euclidean geometry with the modification that ABC implies that A, B and C are all distinct or all identical.

Example 4B. Double elliptic geometry with the modification that AAA' is never true.

The one-dimensional and two-dimensional cases of Postulate 5 are quite dissimilar in substance and might be regarded as separate postulates. If this were done the set Σ_3 would still satisfy all the desired conditions. Independence examples for the one-dimensional case are Examples $\Sigma_1 5$, for the two-dimensional case Examples $\Sigma_1 10$.

* Postulate $\Sigma_1 14A$ follows from this postulate.

DEVELOPMENT.

THEOREM 1.* *If ABC and ACD, BCD.*

Proof. There exists F such that BFD and CFC (p. 5). $F = C$ (p. 3). $\therefore BCD$.

THEOREM 2. *There exist four distinct points.*

Proof. There exist two distinct points A and B (p. 1). There exist points C and D such that $ACB[A \neq C \neq B]$ and $ADC[A \neq D \neq C]$ (p. 2). DCB (th. 1). $\therefore D \neq B$ (p. 3). $\therefore A, B, C$ and D are all distinct.

THEOREM 3. *If A is any point, AAA.*

Proof. There exists B distinct from A (p. 1) and C in the order BCA (p. 2). There exists D in the order BAD (th. 2 and p. 11A or p. 11B). There exists F in the order AFA (p. 5). $F = A$ (p. 3). $\therefore AAA$.

THEOREM 4. *If A and B are any two points, AAB.*

Proof. If $B = A$, AAB (th. 3). If $B \neq A$, there exists C such that ACB and $C \neq B$ (p. 2). $\therefore AAB$ (p. 4).

THEOREM 5.† *If ABC, CBA.*

Proof. ABC and AAB (th. 4). Therefore there exists F such that BFB and CFA (p. 5). $F = B$ (p. 3). $\therefore CBA$.

COROLLARY 1.‡ *If ABC and ABD, then ACD or ADC or $A = B$.*

COROLLARY 2.§ *If A and B are any two points, ABB.*

Since all the one-dimensional postulates in Σ_1 have now been proved, all the further properties of a line follow as from Σ_1. To establish the categoricity of Σ_3 it only remains to prove the following theorem, from which Postulate $\Sigma_1 10$ follows as a corollary.

If $ABC[A \neq B \neq C]$ and $BDE[B \neq D \neq E]$ and E is not on the line AB, then $C \neq D$ and there exists a point F in the orders CDF and AFE.

Proof from $\Sigma_3 A$. $C \neq D$, as in the proof of Theorem $\Sigma_1 15A$.

Suppose that the lines AE and CD have a point F in common. Then F is in one of the orders $(\alpha) FAE$, $(\beta) AFE$, $(\gamma) AEF$ and in one of the orders (a)FCD, (b)CFD, (c)CDF.

(α). If FAE, there exists (since BDE) a point X in the orders FXD and AXB (p. 5). Therefore X is on the lines AB and $FD (= CD)$. $\therefore X = C$. $\therefore ACB$. But ABC. $\therefore B = C$, contrary to hypothesis.

(b). If CFD, there exists (since CBA) a point X in the orders FXA and DXB. Therefore X is on the lines $AF (= AE)$ and $BD (= BE)$. $\therefore X = E$. $\therefore DEB$. But EDB. $\therefore D = E$, contrary to hypothesis.

(γa). If FEA and FCD, there exists a point X in the orders EXD and

* Postulate $\Sigma_1 5$.

† Postulate $\Sigma_1 4$.

‡ Postulate $\Sigma_1 6$.

§ Postulate $\Sigma_1 13B$ follows from Postulate $\Sigma_3 11B$ with the aid of this corollary and Theorems 3 and 4.

AXC. Therefore X is on the lines BE and AB. $\therefore X = B$. $\therefore EBD$. But EDB. $\therefore B = D$ contrary to hypothesis.

(γc). If FEA and FDC, there exists a point X in the orders EXC and AXD. Since ABC and AXD, there exists Y in the orders BYD and CYX. Since CYX and CXE, CYE and Y is on the lines CE and BE. $\therefore Y = E$. $\therefore CEX$ and CXE. $\therefore X = E$. $\therefore AED$. But EDB. $\therefore AEB$, contrary to hypothesis.

Since F is in neither of the orders FAE, AEF, it is in the order AFE. Since ABC and AFE, there exists a point X in the orders BXE and CXF. Since X is on both the lines BE and CD, $X = D$. $\therefore CDF$ and the theorem is true if the lines CD and AE have a point in common.

Suppose they have no point in common. There exists a point G distinct from A in the order CAG. Since the distinct lines AB and AE have only A in common, $GE \neq AE$. Moreover GE and AE are both in the plane ABE, in which CD lies. Therefore the lines GE and CD have a point X in common (p. 12A). As before, GXE. Since also GAC, there exists a point F in the orders XFC and EFA. Therefore the lines CD and AE have a point in common, contrary to supposition. The theorem is therefore true.

Proof from $\Sigma_3 B$. $C \neq D$ as before. BAC' and BDE. Therefore there exists a point F in the orders AFE and $C'FD$. Since $C'FD$, CDF.

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FURTHER PROPERTIES OF THE GENERAL INTEGRAL.

By P. J. Daniell.

Introduction. In these Annals there appeared a paper by the author on a general form of integral of which the Lebesgue integral and the Radon-Young integral are special cases.* That paper confined itself more particularly to the definitions and proofs of existence. In this paper some special points are discussed which bring the theory into closer relation with the Lebesgue theory. It is advisable first to give a resumé of the foundations of our general theory.

We consider functions $f(p)$ of general elements, p. A class T_0 of such functions is assumed which is closed with respect to the operations, multiplication by a constant, addition and taking the modulus. It is also assumed that to each f of class T_0 there corresponds a number K, independent of p, such that

$$|f(p)| \leq K,$$

and that to each f there corresponds a finite "integral" $S(f)$ possessing the properties C, A, L, M, or an integral $I(f)$ having the properties C, A, L, P. If $U(f)$ is a functional operation on f these properties are

(C) $$U(cf) = cU(f),$$

(A) $$U(f_1 + f_2) = U(f_1) + U(f_2).$$

(L) If $f_1 \geq f_2 \geq \cdots \geq 0 = \lim f_n$, then

$$\lim U(f_n) = 0.$$

(P) $$U(f) \geq 0 \quad \text{if} \quad f \leq 0.$$

(M) $$|U(f)| \leq M(|f|).$$

It will be necessary to make frequent references to the earlier paper and these will be given by D 3(1), for example. We shall also use a notation which was used in that paper

$$f(p) = f_1(p) \vee f_2(p),$$

the logical sum of f_1, f_2, if, for each p, the value of f is the greater of the

* P. J. Daniell, Annals of Mathematics, vol. 19 (1918), p. 279.
J. Radon, Sitzungsberichte der Akademie der Wissenschaften, Wien (1913), p. 1295.
W. H. Young, Proceedings of the London Mathematical Society, vol. 13 (1914), p. 109.

values of f_1, f_2. If the lesser value is chosen the combination is called the logical product

$$f = f_1 \wedge f_2.$$

One particular theorem is so valuable that we restate it here for the convenience of the reader.

D. 7(7). If $f_1, f_2, \cdots$ is a sequence of summable functions with limit f, and if a summable function φ exists such that $|f_n| \leq \varphi$ for all n, f is summable, lim $I(f_n)$ exists and $= I(f)$.

In the first paragraph we prove that an S-integral can be expressed in terms of a single I-integral, and use the concept of convergence in the mean. In the second paragraph measurable functions are defined, and it is proved that, under certain conditions, the fourth proportional of three measurable functions is measurable. The third paragraph deals with repeated integration and sequences of integrals. Finally we apply the previous analysis to special cases, more particularly to the generalized Stieltjes integral. Those who are more interested in the latter may turn immediately to the end of the paper.

1. **The S-integral as an I-integral.** In the previous paper D. 3. we defined $I_1(f)$ as the upper bound of $S(\varphi)$ for all functions φ of class T_0 such that $0 \leq \varphi \leq f$. $I_1(f)$ is called the positive integral, $I_2(f) = I_1(f) - S(f)$ the negative and $I(f) = I_1 + I_2$ the modular integral associated with S. Then $S(f) = I_1(f) - I_2(f)$, the difference of two I-integrals. Now in theorem 1(7) we prove that a function λ, equal to 1 or -1 everywhere, exists such that

$$S(f) = I(\lambda f).$$

1(1). Definition. A sequence of functions, f_n, is *convergent in the mean* with respect to I if the functions, f_n, are summable I and if

$$\lim_{m, n \to \infty} I(|fm - f_n|) = 0.$$

1(2). Definition. A sequence of functions, f_n, *converges in the mean to* f if f_n, f are summable I, and if

$$\lim_{n \to \infty} I(|f - f_n|) = 0.$$

These two definitions can be proved to be equivalent; for if the conditions of 1(2) are satisfied, given any $e > 0$ we can find n_0 so that

$$I(|f_m - f_n|) \leq I(|f - f_m|) + I(|f - f_n|)$$
$$< e + e \qquad m, n \geq n_0.$$

The following theorem proves the converse.

1(3). If the sequence, f_n, is *convergent in the mean* there exists a summable function f to which the sequence converges in the mean. Given any $e > 0$ we can find n_0 so that

$$I(|f_m - f_n|) < e \quad m, n \geqq n_0.$$

Let $n_1, n_2, \cdots, n_s, \cdots$ be the numbers corresponding to e equal to 2^{-1}, $2^{-2}, \cdots, 2^{-s}, \cdots$ but picking them out so that $n_1 < n_2 < \cdots$. Then

$$I(|f_m - f_n|) < 2^{-s} \quad m, n \geqq n_s.$$

Let $f_{n_s} = g_s$ and

$$\varphi_{st} = g_s \vee g_{s+1} \vee \cdots \vee g_{s+t}.$$

At a particular p, φ_{st} will be one of the values of g_{s+r}, $r = 0, \ , \cdots t$. But

$$\begin{aligned} g_{s+r} &= g_s + (g_{s+1} - g_s) + \cdots + (g_{s+r} - g_{s+r-1}) \\ &\leqq g_s + |g_{s+1} - g_s| + \cdots + |g_{s+r} - g_{s+r-1}|. \end{aligned}$$

If we denote

$$\begin{aligned} \psi_{st} &= g_s + |g_{s+1} - g_s| + \cdots + |g_{s+t} - g_{s+t-1}|, \\ \varphi_{st} &\leqq \psi_{st}, \\ I(\varphi_{st}) &\leqq I(\psi_{st}) \\ &< I(g_s) + 2^{-s} + 2^{-s-1} + \cdots + 2^{-s-t+1} \\ &< I(g_s) + 2^{-s+1}. \end{aligned}$$

$\varphi_{s1} \leqq \varphi_{s2} \leqq \cdots$ is a nondecreasing sequence so that by D7(6) $\lim\limits_{t \doteq \infty} \varphi_{st} = \varphi_s$ exists, is summable I and

$$I(\varphi_s) \leqq I(g_s) + 2^{-s+1}.$$

Again

$$\begin{aligned} -\varphi_s \leqq \varphi_{st} &\leqq - g_s \\ &= -g_1 + (g_1 - g_2) + \cdots + (g_{s-1} - g_s) \\ &\leqq -g_1 + |g_1 - g_2| + \cdots + |g_{s-1} - g_s|. \\ I(-\varphi_s) &\leqq I(-g_1) + 2^{-1} + 2^{-2} + \cdots + 2^{-s+1} \\ &\leqq I(-g_1) + 1. \end{aligned}$$

But $-\varphi_1 \leqq -\varphi_2 \leqq \cdots$ is nondecreasing and by the same theorem $\lim\limits_{s \doteq \infty} (-\varphi_s) = -f$ exists, is summable and

$$I(f) = \lim_{s \doteq \infty} I(\varphi_s).$$

Given any positive e we can find s so that both $2^{-s+2} < e$ and

$$I(|f - \varphi_s|) < \tfrac{1}{4}e.$$

Then because $\varphi_s \geqq g_s$

$$I(|\varphi_s - g_s|) = I(\varphi_s) - I(g_s)$$
$$< 2^{-s+1} < \tfrac{1}{2}e.$$

If n_0 is the number n_s corresponding to the above s,

$$I(|g_s - f_n|) < 2^{-s} < \tfrac{1}{4}e \qquad n \geqq n_0.$$

Combining the three inequalities

$$I(|f - f_n|) < e \qquad n \geqq n_0.$$

This proves the theorem and therefore the definitions are equivalent.

Definition. Two functions are *nearly equal* with respect to I if the integral of their modular difference is zero. If f, g are two functions to which a sequence converges in the mean they must be "nearly equal," for given $e > 0$ we can find n_0, n_1 so that

$$I(|f - g|) \leqq I(|f - f_n|) + I(|g - f_n|)$$
$$< 2e \qquad n \geqq n_0 / n_1.$$

1 (4) Definition. The *outer* measure of a set e of elements p is defined as the upper semi-integral $\bar{I}(f)$ of the function equal to 1 on e and 0 otherwise. Since the above function is non-negative its lower semi-integral is non-negative; so that, if a set has a zero outer measure the corresponding function is summable and has the integral 0. In this case we say that the set is of zero measure.

Theorem. Two nearly equal functions are equal *nearly everywhere*, that is, except on a set of zero measure. For if their modular difference be called h, $h \geqq 0$ and $I(h) = 0$. If $E(e)$ is the set where $h \geqq e > 0$ and f_e the corresponding function, $ef_e \leqq h$ and $e\bar{I}(f_e) \leqq I(h) = 0$ by D. 6(3). Then $\underline{I}(f_e) \leqq \bar{I}(f_e) = 0 = I(f_e)$. Now as e approaches zero monotonously the sequence f_e is nondecreasing and of constant zero integral. If E is the set where $h > 0$, $f_E = \lim_{e=0} f_e$ and the measure of $E = I(f_E) = \lim I(f_e) = 0$.

1 (5). If $h \geqq 0$ is of class T_0 there exists a function k summable I such that $0 \leqq k \leqq h$ and

$$S(k) = I_1(h).$$

Since $I_1(h)$ is by definition the upper bound of $S(\varphi)$ $0 \leqq \varphi \leqq h$ given $e_i > 0$ we can find φ_i such that $0 \leqq \varphi_i \leqq h$ and

$$I_1(\varphi_i) - I_2(\varphi_i) = S(\varphi_i) > I_1(h) - e_i$$
$$I_1(h - \varphi_i) + I_2(\varphi_i) < e_i.$$

Similarly given $e_j > 0$ we can find φ_j so that

$$I_1(h - \varphi_j) + I_2(\varphi_j) < e_j.$$

But $|\varphi_i - \varphi_j|$ is not greater than $\varphi_i + \varphi_j$ and also not greater than $|h - \varphi_i| + |h - \varphi_j|$ so that

$$\begin{aligned} I(|\varphi_i - \varphi_j|) &= I_1(|\varphi_i - \varphi_j|) + I_2(|\varphi_i - \varphi_j|) \\ &< e_i + e_j. \end{aligned}$$

If $e_i \doteq 0$, $e_j \doteq 0$ as $i, j \doteq \infty$

$$\lim_{i,j \doteq \infty} I(|\varphi_i - \varphi_j)| = 0.$$

By definition 1(1), the sequence, φ_i, is convergent in the mean and by theorem 1(3) it converges in the mean to a summable function k. Recalling the method by which k is found and since $0 \leq \varphi_i \leq h$, $0 \leq k \leq h$ and

$$\begin{aligned} I_1(h - k) + I_2(k) &= \lim_{i \doteq \infty} I_1(h - \varphi_i) + I_2(\varphi_i) \\ &= 0. \end{aligned}$$

But both $I_1(h - k)$ and $I_2(k)$ are non-negative so that each is zero separately, and

$$S(k) = I_1(k) - I_2(k) = I_1(h) - 0 = I_1(h).$$

Since $0 \leq k \leq h$, k vanishes with h and we can find a function θ such that $0 \leq \theta \leq 1$ and $k = \theta h$. Then

$$S(\theta h) = I_1(h).$$

But $I_1(h - \theta h) = 0$, $I_2(\theta h) = 0$ so that $\theta = 1$ except on a set of I_1 — measure 0 and $\theta = 0$ except on a set of I_2 — measure 0; or $\theta = 0$ or 1 except on a set of I-measure 0.

1(6). If f is summable and vanishes with h

$$S(\theta f) = I_1(f).$$

It is sufficient to assume f non-negative for otherwise it is the difference of two non-negative functions.

If

$$f_n = f \wedge nh, \qquad I_1(f) = \lim I_1(f_n), \qquad S(\theta f) = \lim S(\theta f_n).$$

But

$$I_1(f_n) - I_1(\theta f_n) \leq nI_1(h - k) = 0$$

$$I_2(\theta f_n) \leq nI_2(k) = 0.$$

$$S(\theta f) = \lim S(\theta f_n) = \lim I_1(f_n) = I_1(f).$$

1(7). If $h \geq 0$ is summable S and limited there exists a function λ

which is everywhere equal to 1 or -1 and such that if f is summable and vanishes with h,

$$S(f) = I(\lambda f).$$

Extend the class T_0 to include h and combinations of h with members of T_0 so that the new class $T_0(+h)$ satisfies the required conditions for a class T_0. In obtaining $I_1(f)$ as the upper bound of $S(\varphi)$ we have to consider at least all the old functions φ with some additions. Then the new $I_1(f)$ $f \geq 0$ cannot be less than the old. On the other hand for all the new $\varphi \leq f$ as well as the old, $S(\varphi)$ is not greater than the old $I_1(f)$. The old and the new positive integrals and consequently the old and new modular integrals will be the same. Then all the way through the integrals will be identical. But in the new class $T_0(+h)$, h is a member and the previous theorems can be applied. Define $\theta' = \theta$ when $\theta = 1$ or 0 and $\theta' = 1$ otherwise, that is on a set of zero measure. Then $\theta' f$ is summable and

$$S(\theta' f) = I_1(f).$$

Let $\lambda = 2\theta' - 1 = 1$ or -1 everywhere. Then λf is summable and

$$\begin{aligned} f &= \lambda^2 f = 2\theta'\lambda f - \lambda f, \\ S(f) &= S(2\theta'\lambda f) - S(\lambda f) \\ &= 2I_1(\lambda f) - S(\lambda f) \\ &= I(\lambda f). \end{aligned}$$

If it is possible to find a function $h > 0$ summable S this theorem would be true for all functions summable S. But in the general theory there may be no such function h vanishing nowhere and summable. We return to this question in the last paragraph.

2. **Measurable functions.** In the theory of Lebesgue integrals there is a valuable distinction between summable and measurable functions. In the general integral it is equally valuable and any function belonging to the Borel-extension of T_0 (extension by successive limiting processes and linear combinations) is measurable.

Definition. If $h \geq 0$ is summable I and nowhere infinite, f is said to be measurable hI if the function $mh \vee f \wedge Mh$ is summable for all m, M $(m \leq M)$. This function is equal to mh when $f < mh$, to f when $mh \leq f \leq Mh$ and to Mh when $f > Mh$. Any summable function is measurable hI for every h.

$$mh \vee (f \vee g) \wedge Mh = (mh \vee f \wedge Mh) \vee (mh \vee g \wedge Mh)$$

so that by using $D7(5)$ the logical sum, and similarly the logical prdouct,

of any two functions measurable hI is measurable hI. In particular if f is measurable so are $f \vee 0$, $-f \vee 0$. The converse is also true, that if $f \vee 0$, $-f \vee 0$ are measurable so is f. Define the function

$$\begin{aligned}\varphi(p, s, f) &= 0 \quad f(p) \leq sh(p)\\ &= h(p) \quad f(p) > sh(p).\end{aligned}$$

2(1). The necessary and sufficient condition that f be measurable hI is that $\varphi(p, s, f)$ is summable for all s.

It is necessary, for if f is measurable hI, the function $f - sh$ is also measurable and $\varphi(p, s, f) = \varphi(p, 0, f - sh)$. We only require to prove that if f is measurable hI, $\varphi(p, 0, f)$ is summable. Also $f \vee 0$ is measurable and $\varphi(p, 0, f) = \varphi(p, 0, f \vee 0)$, and therefore we can assume f non-negative without loss of generality. Let $\varphi_n = nf \wedge h$ then φ_n is summable and since $nf \leq (n + 1)f$ (f being non-negative), $\varphi_1 \leq \varphi_2 \leq \cdots$ is a nondecreasing sequence of summable functions and $|\varphi_n| = \varphi_n \leq h$ which is summable. If at $p, f = 0$, $\varphi_n = 0$ $(n = 1, 2, \cdots)$ or $\lim \varphi_n = 0$. On the other hand, if at $p, f > 0$, some n_0 exists depending possibly on p such that $nf > h$, $n \geq n_0$ and then $\lim \varphi_n = h$. Therefore $\lim \varphi_n = \varphi(p, 0, f)$ is summable by D.7(7).

The condition is sufficient for considered as a function of s, $\varphi(p, s, f)$ is non-increasing and limited for each separate p. Consequently φ is integrable Riemann with respect to s in a limited interval. Consider the Riemann integral

$$\int_m^M \varphi(p, s, f)ds.$$

If $s_0 = m < s_1 < \cdots < s_n = M$ and if $s_i \leq t_i < s_{i+1}$, the integral is defined as the limit of

$$\sum_i \varphi(p, t_i, f)(s_{i+1} - s_i),$$

as the maximum difference $s_{i+1} - s_i$ approaches zero. The above sum is a linear combination of summable functions and its modulus cannot exceed $(M - m)\varphi(p, m, f)$ which is summable so that by $D.7(7)$ the limit function, that is, the function of p defined by the Riemann integral is summable. But this integral is exactly the function

$$mh \vee f \wedge Mh - mh.$$

Case 1. $h(p) = 0$, $\varphi(p, s, f) = 0$ for all s, the integral $= 0$ and so is the function designated.

Case 2. $f(p) > Mh(p)$, $\varphi(p, s, f) = h$ $(m \leq s \leq M)$ the integral $= Mh - mh$.

Case 3. $f(p) \leqq mh(p)$, $\varphi(p, s, f) = 0$ $(m \leqq s \leqq M)$ and the integral $= 0 = mh - mh$.

Case 4. $f(p) = th(p)$, $m < t \leqq M$ then $\varphi(p, s, f)$ equals h when $s < t$ and 0 beyond so that the integral becomes $(t - m)h = f - mh$. Then the function designated is equal to that defined by the integral and is summable; or $mh \vee f \wedge Mh$ is summable and f is measurable hI. If we make $m = 0$, we obtain at the same time the relation

$$f \vee 0 = \int_0^\infty \varphi(p, s, f)ds,$$

provided f vanishes with h. For then

$$\begin{aligned} 0 \vee f &= \lim_{M \to \infty} 0 \vee f \wedge Mh \\ &= \lim_{M \to \infty} \int_0^M \varphi(p, s, f)ds \\ &= \int_0^\infty \varphi(p, s, f)ds \end{aligned}$$

in the sense that when $0 \vee f$ is finite the equality is valid while if it is infinite, the integral is divergent to $+ \infty$. The same theorem which proves that the above Riemann integral exists proves that if $f \geqq 0$ is summable and vanishes with h,

$$I(f) = \int_0^\infty I[\varphi(p, s, f)]ds.$$

2(2). Definition. If f is measurable hI and if a function $\theta(p)$ exists, finite for each p, such that $f = \theta h$, f is said to be *commensurable* hI. If f satisfies this condition and is also summable it is said to be *summable* hI.

Theorem. If $f \geqq 0$ is commensurable hI, either $\underline{I}(f) = \infty$ or f is summable hI.

For $f \wedge nh = f_n$ is summable hI and $f_1 \leqq f_2 \leqq \cdots \leqq f = \lim f_n$. If then $\underline{I}(f)$ is finite, $I(f_n) \leqq \underline{I}(f)$ so that by $D.7(6)$, $f = \lim f_n$ is summable.

In the following theorems of this paragraph there will enter some ratios of functions. We shall use the convention that *a meaningless fraction* of the form $\frac{0}{0}$ *is replaced by* 0.

2(3). If $f \geqq 0$ is commensurable hI and λ is any constant index, $h(f/h)^\lambda$ is commensurable hI where the above convention is used and where in the case of several values the real non-negative value is chosen. Denote the function by g and $1/\lambda$ by μ. If $\lambda > 0$, when $g > sh$, $f > s^\mu h$

and when $g \leq sh$, $f \leq s^\mu h$ so that

$$\varphi(p, s, g) = \varphi(p, s^\mu, f)$$

and this is summable. Then by theorem 2(1) g is measurable hI.

$\varphi(p, s, f)$ is a monotone function of s, summable I and of modulus not greater than h. Hence

$$\varphi(p, s - 0, f) = \lim_{\epsilon \doteq 0,\, \epsilon > 0} \varphi(p, s - \epsilon, f)$$

is summable. But if $\lambda < 0$,

$$\varphi(p, s, g) = h(p) - \varphi(p, s^\mu - 0, f)$$

is summable. If $\lambda = 0$, $g = h$ is measurable hI. Again, if $f = \theta h$, $g = \theta^\lambda h$ so that g is commensurable hI.

2(4). If f, g are commensurable hI, so is $f + g$. The functions

$$\begin{aligned} \varphi(p, s - 0, g) &= 0 \quad g < sh \\ &= h \quad g \geq sh \end{aligned}$$

and $\varphi(p, s + 0, g) = \varphi(p, s, g)$ are summable. The product of two functions of type φ divided by h is also their logical product,

$$\varphi_1\varphi_2/h = \varphi_1 \wedge \varphi_2$$

and is summable. Also the function $\varphi(p, t, g)/h$ is a bounded non-increasing function of t and the generalized Stieltjes integral

$$-\int_{-\infty}^{+\infty} \varphi(p, s - t, f)d_t\varphi(p, t, g)/h$$

exists and the function so defined is summable I. The proof uses $D.7(7)$ as in theorem 2(1). At a particular p suppose that $g = t_1h$, then $\varphi(p, t, g)/h = 0$ if $t > t_1$ and 1 if $t \leq t_1$. Then the value of the integral (with the minus sign) will be

$$\varphi(p, s - t_1, f)[\varphi(p, t_1 - 0, g)/h - \varphi(p, t_1 + 0, g)/h] = \varphi(p, s - t_1, f).$$

That is to say, the function defined by the integral will be 0 or h according to whether $f \leq (s - t_1)h$ or $> (s - t_1)h$, or whether $f + g \leq sh$ or $f + g > sh$. Therefore the integral represents $\varphi(p, s, f + g)$ which must be summable from what was said before about the integral. Using theorem 2(1), $f + g$ is measurable hI. Also if $f = \theta_1h$, $g = \theta_2h$, $f + g = (\theta_1 + \theta_2)h$ and is commensurable hI.

Corollary. Any linear combination of functions commensurable hI is commensurable hI.

2(5). If f, g are commensurable hI so is fg/h, the fourth proportional

of f, g and h. In succession using 2(4) Cor. and 2(1) we can show that $k = \frac{1}{2}(f + g)$, $l = \frac{1}{2}(f - g)$, $k \vee 0$, $-k \vee 0$, $l \vee 0$, $-l \vee 0$, and $h(k/h)^2 = h(k \vee 0)^2/h^2 + h(-k \vee 0)^2/h^2$, $h(l/h)^2$ are commensurable hI. Finally

$$fg/h = h(k/h)^2 - h(l/h)^2$$

is commensurable hI.

2(6). The limit of a convergent sequence of functions measurable hI is measurable hI. For if $f = \lim f_n$, at each element p

$$mh \vee f \wedge Mh = \lim (mh \vee f_n \wedge Mh).$$

$$|mh \vee f_n \wedge Mh| \leqq (|M| + |m|)h$$

which is summable; or, by D7(7), $mh \vee f \wedge Mh$ is summable and f is measurable hI.

Combining this theorem with 2(4) Cor., any function belonging to the Borel extension of the subclass of functions of T_0 commensurable hI, is also commensurable hI.

2(7). If f is measurable hI, it is measurable kI for any function $k \geqq 0$ which is nowhere infinite and which vanishes with h. The function

$$\begin{aligned} \varphi(p, s, f) &= 0 \quad f \leqq sh \\ &= h \quad f > sh, \end{aligned}$$

is summable and is a non-increasing bounded function of s. The same is true when k takes the place of f. The integral

$$-\int_{m-0}^{M+0} t\varphi(p, st, f)d_t\varphi(p, t, k)/h$$

exists and can be proved to be summable by a method similar to that used in the proof of theorem 2(1). If at a certain p, $k = ch$, $m \leqq c \leqq M$, the negative integral has the value

$$\begin{aligned} c\varphi(p, sc, f) &= 0 \qquad\qquad f \leqq sch = sk \\ &= ch = k \quad f > sch = sk. \end{aligned}$$

On the other hand if c lies outside the range mM, $\varphi(p, t, k)/h$ is constant throughout the range and the integral is zero. The function defined by the integral is summable I for all m, M and in modular value does not exceed k which is summable. As a function of m or M it is monotone so that the limit as $m \doteq -\infty$, $M \doteq +\infty$ exists and is summable I. This limit is the function equal to 0 when $f \leqq sk$ and to k when $f > sk$, or it is the function $\varphi_1(p, s, f)$ using k as a base function in place of h. The theorem is therefore proved.

2(8) If f, g, k are commensurable hI is fg/k. For by 2(3), h^2/k is commensurable hI and by 2(5) fg/h is also. But

$$fg/k = (fg/h)(h^2/k)/h.$$

2(9) Define the function u equal to 1 when $h > 0$ and to 0 when $h = 0$. *If the function equal to 1 everywhere is measurable hI, u is commensurable hI.* Then h^λ is commensurable hI where λ is a constant index; for

$$h^\lambda = h(u/h)^{1-\lambda}.$$

Also if f, g are commensurable hI so also are f/g, fg; for

$$f/g = uf/g,$$

$$fg = fg/u.$$

3. **Iterated integrals and sequences of integrals.** 3(1). Let $S(q, f)$ be a collection of S-integrals on functions f of elements p, distinguished by the letters q which also refer to general elements not necessarily the same as p. Let these S-integrals be defined in terms of the same initial class T_0 of functions of p. Let Q be a positive I-integral on functions of q such that if f belongs to T_0, $S(q, f)$, $I(q, f)$ the modular integral associated with $S(q, f)$ are summable Q. For such functions f define

$$S(f) = Q[S(q, f)].$$

Then this equality holds for any function f belonging to the Borel extension of T_0 provided $I(q, |f|)$ is summable Q.

As at the beginning of paragraph 2 by the Borel extension is meant the extension by successive linear combinations and limiting processes. It is evident that the $S(f)$ so defined satisfies (C) and (A). Also

$$\begin{aligned} |S(f)| &\leq Q[|S(q, f)|] \\ &\leq Q[I(q, |f|)] \end{aligned}$$

so that the condition M is satisfied by S. Again if

$$f_1 \geq f_2 \geq \cdots \geq 0 = \lim f_n,$$

$I(q, f_n)$ is in modular value not greater than $I(q, f_1)$ which is summable Q. It follows from D7(7) that

$$\begin{aligned} |\lim S(f_n)| &\leq \lim Q[I(q, f_n)] \\ &= Q[\lim I(q, f_n)] \\ &= Q(0) = 0. \end{aligned}$$

Therefore $S(f)$ satisfies all the required conditions for an S-integral.

If $0 \leq \varphi \leq f$,

$$\begin{aligned} S(\varphi) &= Q[S(q, \varphi)] \\ &\leq Q[I_1(q, \varphi)] \\ &\leq Q[I_1(q, f)], \end{aligned}$$

and therefore the upper bound $I_1(f)$ satisfies

$$I_1(f) \leq Q[I_1(q, f)]$$

$$I_2(f) = I_1(f) - S(f) \leq Q[I_2(q, f)]$$

provided $f \geq 0$.

Now let f be a function of class T_1, the limit of a non-decreasing sequence, f_n, of functions of class T_0, such that $I(q, f)$ is summable Q, then since $|f_n| \leq |f| + |f_1|$,

$$\begin{aligned} I_1(f_n) &\leq Q[I_1(q, |f_n|)] \\ &\leq Q[I_1(q, |f|)] + Q[I_1(q, |f_1|)] \\ &\leq Q[I(q, |f|)] + Q[I(q, |f_1|)] \end{aligned}$$

which is finite. Then by $D7(6)$, $f = \lim f_n$ is summable I_1 and $I_1(f) = \lim I_1(f_n)$. Similarly $I_2(f) = \lim I_2(f_n)$, or f is summable S and

$$S(f) = \lim S(f_n) = \lim Q[S(q, f_n)].$$

But

$$|S(q, f_n)| \leq I(q, |f_n|) \leq I(q, |f|) + I(q, |f_1|)$$

which is summable Q, so that by D7(7)

$$\begin{aligned} S(f) &= Q[\lim S(q, f_n)] \\ &= Q[S(q, f)]. \end{aligned}$$

Evidently any linear combination of such functions f will again satisfy the equality and a similar proof may be used successively for one monotone limiting process after another, with linear combinations interpolated. Consider a more general limiting process $f = \lim f_n$ where the functions f_n already satisfy the equality, and where $I(q, |f|)$ is summable Q. We can choose f and f_n to be non-negative without loss of generality for $f \vee 0 = \lim (f_n \vee 0)$, $-f \vee 0 = \lim (-f_n \vee 0)$, $f = f \vee 0 - (-f \vee 0)$. Since $f = \lim f_n = \lim f_n$, f is the limit of a non-decreasing sequence $g_1 \leq g_2 \leq \cdots \leq f$ where $g_n = f_n \wedge f_{n+1} \wedge \cdots$ is the limit of a non-increasing sequence, but all the functions involved lie between 0 and f inclusive so that proofs similar to that used above may be used to show that

$$S(g_n) = Q[S(q, g_n)],$$

$$S(f) = Q[S(q, f)],$$

in succession.

Before considering sequences of integrals we consider an operation which is a slight modification of the Moore general integral of which the limit of a sequence is a special case.

3(2). DEFINITION. $R(u)$, a functional operation on functions u of the elements q is said to be a *Moore positive integral* if there exists a class U of functions u which is equal to its star-extension,* which includes the modulus of each member of the class and such that for members of U the operation R satisfies

(A) $$R(u_1 + u_2) = R(u_1) + R(u_2),$$

(C) $$R(cu) = cR(u),$$

(P) $$R(u) \geq 0 \text{ if } u \geq 0.$$

If $u_1 \leq u_2$, $R(u_1) \leq R(u_2)$ since $R(u_2 - u_1) \geq 0$. If

$$\lim u_n = u\,([q];\ U),$$

$$\lim R(u_n) = R(u).$$

For if the sequence, u_n, converges to u relative to the class U, that means that there exists a member $v(q)$ of U which we suppose replaced by its modulus such that for every $e > 0$ there is an n_e such that

$$|u - u_n| \leq ev \qquad n \geq n_e.$$

Then

$$\begin{aligned} |R(u) - R(u_n)| &= |R(u - u_n)| \\ &\leq R(|u - u_n|) \\ &\leq eR(v). \end{aligned}$$

3(3). DEFINITION. The collection $S(q, f)$ of S-integrals on functions f of the elements p and distinguished by the elements q is said to be *dominated by* $J(f)$ *with relative uniformity in* $R(u)$ where R is a Moore positive integral if $S(q, f)$, J are defined in terms of the same class T_0 of functions of p, if for members of T_0, $S(q, f)$ belongs to the class U of functions of q, and if there exists a function $v(q)$ belonging to U, independent of f such that for each $f \geq 0$ of class T_0,

$$I(q, f) \leq v(q)J(f).$$

Here I is the modular integral associated with S, $J(f)$ is an I-integral. Corresponding to this definition we may say that a single S-integral $S(f)$ is dominated by $J(f)$ if they possess the same class T_0 and if a number M exists independent of f such that if $f \geq 0$ belongs to T_0,

$$I(f) \leq MJ(f),$$

when I is the modular integral associated with S.

* E. H. Moore, Bulletin of the American Mathematical Society, vol. 18 (1912), p. 345.

3(4). **Theorem.** Under the conditions of 3(3)

$$S(f) = R[S(q, f)]$$

is dominated by J and the equality holds for all functions f summable J. Define the class $T_0(S)$ to be the class $T_0(J)$, then it is seen that for functions belonging to T_0, $S(f_1 + f_2) = S(f_1) + S(f_2)$, $S(cf) = cS(f)$ and $|S(f)| \leq R(v)J(|f|)$. If $f \geq 0$, $-f \leq \varphi \leq +f$.

$$S(\varphi) \leq R(v)J(|\varphi|) \leq R(v)J(f).$$

But $I(f)$ is the upper bound of such $S(\varphi)$ so that

$$I(f) \leq R(v)J(f).$$

Therefore $S(f)$ is dominated by J. If $f_1 \geq f_2 \geq \cdots \geq 0 = \lim f_n$,

$$\lim |S(f_n)| \leq R(v) \lim J(f_n) = 0.$$

Therefore $S(f)$ satisfies all the required properties of an S-integral. S is dominated by J, any function summable J is summable S and the dominance inequality remains valid. For we can prove this first for f a member of T_1, summable J, and from this that if $f \geq 0$ is summable J

$$\dot{I}(f) \leq MJ(f).$$

But given $e > 0$ and a function f summable J we can find a member f of T_0 such that

$$J(|f - f_e|) < e/M.$$

$$\dot{I}(|f - f_e|) < e$$

so that f is summable I.

In this particular case $M = R(v)$,

$$\begin{aligned} |S(f) - S(f_e)| &\leq I(|f - f_e|) \\ &\leq R(v)J(|f - f_e|) \\ &< e, \\ |R[S(q,f)] - R[S(q,f_e)]| &\leq R[I(q, |f - f_e|)] \\ &\leq R(v)J(|f - f_e|) \\ &< e, \\ S(f_e) &= R[S(q, f_e)]. \end{aligned}$$

Therefore $S(f)$ differs from $R[S(q, f)]$ by less than $2e$ which may be as small as we please and consequently

$$S(f) = R[S(q, f)].$$

3(5). If a sequence, $S_n(f)$, of S-integrals is *uniformly dominated* by J and if, for every member of $T_0(J)$, $S_n(f)$ approaches a limit, this limit is an S-integral $S(f)$ and for every f summable J

$$S(f) = \lim S(f_n).$$

By "uniformly dominated" we mean that each S_n is dominated by J and that the corresponding numbers M_n are limited in their set.

This theorem can be regarded as a special case of 3(4). Take the numbers, n, to be the elements q, the class U to be the class of functions u_n for which $\lim u_n$ exists and let

$$R[u(n)] = \lim_{n \doteq \infty} u_n.$$

The function $v(q)$ will be taken to be a constant and equal to the upper bound of the numbers M_n. Then $R(v)$ = this upper bound.

4. **The Stieltjes integral and other examples.** We shall merely state some of the foregoing definitions and theorems as they are applicable to generalized Stieltjes (Radon-Young) integrals. Each is denoted by the letter a added to the number of the corresponding definition or theorem of which it is an application. Other applications can be supplied by the reader.

1(7a). If $\alpha(x, y)$ is of limited two-dimensional variation in $a \leq x \leq b$, $c \leq y \leq d$, and if $\omega(x, y)$ is its corresponding variation function, a functon $\lambda(x, y)$ = either 1 or -1 everywhere exists such that

$$\int_{a,c}^{b,d} f(x, y)d\alpha(x, y) = \int_{a,c}^{b,d} f(x, y)\lambda(x, y)d\omega(x, y)$$

for all functions $f(x, y)$ summable (ω). To apply theorem 1(7) choose $h(x, y) \equiv 1$. The one-dimensional case has been stated already by the author.*

3(1a). Let $\beta(x)$ be a limited non-decreasing function $a \leq x \leq b$; let $\alpha(x, s)$ be of limited variation in $s(c \leq s \leq d)$, the total variation being uniformly limited in x $(a \leq x \leq b)$, and for each s let it be limited and measurable Borel $(a \leq x \leq b)$. Then

$$\int_c^d f(s)d_s \int_a^b \alpha(x, s)d\beta(x) = \int_a^b \left[\int_c^d f(s)d_s\alpha(x, s) \right] d\beta(x),$$

provided $f(s)$ is limited and measurable Borel $c \leq s \leq d$. To apply 3(1) we choose $p = s$, $q = x$ and the functions of class T_0 to be the step-functions (constant over each of a finite set of subintervals into which (cd) is

* P. J. Daniell, Transactions of the American Mathematical Society, vol. 19 (1918), p. 361.

divided). Such functions are linear combinations of functions of the type

$$f(s) = 1 \quad s_1 \leq s < s_2$$
$$= 0 \quad \text{otherwise}$$

and for this

$$\int_c^d f(s)d_s \int_a^b \alpha(x, s)d\beta(x) = \int_a^b \alpha(x, s_2 - 0)d\beta(x) - \int_a^b \alpha(x, s_1 - 0)d\beta(x)$$
$$= \int_a^b \left[\int_c^d f(s)d_s\alpha(x, s)\right]d\beta(x).$$

Again if $f(s)$ is limited and measurable Borel,

$$I(q, |f|) = \int_c^d |f(s)|\, d_s\omega(x, s)$$

where ω is the variation function corresponding to $\alpha(c \leq s \leq d)$

$$\leq T \max |f(s)|,$$

where T is the upper bound of the total variations of $\alpha(x, s)$, and since $\omega(x, s)$ is also measurable Borel $(a \leq x \leq b)$, $I(q, |f|)$ is summable β.

Corollary. This theorem can be immediately extended by allowing $\beta(x)$ to be a function of limited variation $(a \leq x \leq b)$. So extended the theorem is a generalization of one proved by Bray.*

3(5a). We say that a sequence, $\alpha_n(x)$ is uniformly Δ-bounded by a non-decreasing function $\gamma(x)$ if a number M exists independent of n, x_1, x_2 such that

$$|\alpha_n(x_2) - \alpha_n(x_1)| \leq M[\gamma(x_2) - \gamma(x_1)] \quad a \leq x_1 \leq x_2 \leq b.$$

Then the sequence of Stieltjes integrals,

$$\int_a^b f(x)d\alpha_n(x)$$

is uniformly dominated by $\int_a^b f(x)d\gamma(x)$. If the sequence $\alpha_n(x)$ is uniformly Δ-bounded by $\gamma(x)$ and if for each x it converges to $\alpha(x)$, $\alpha(x)$ is Δ-bounded by $\gamma(x)$ and

$$\int_a^b f(x)d\alpha(x) = \lim_{n \doteq \infty} \int_a^b f(x)d\alpha_n(x),$$

provided $f(x)$ is summable γ.

* H. E. Bray, Annals of Mathematics, vol. 20 (1919), p. 183.

To apply 3(5) we choose J as the integral with respect to γ, T_0 the class of step-functions and since these are linear combinations of functions of the type

$$f(x) = 1 \quad a \leq x_1 \leq x < x_2 \leq b$$
$$= 0 \quad \text{otherwise}$$

it is sufficient to prove that

$$\lim \left[\alpha_n(x_2 - 0) - \alpha_n(x_1 - 0)\right] = \alpha(x_2 - 0) - \alpha(x_1 - 0).$$

Given $e > 0$ we can find $\delta > 0$ so that

$$\gamma(x_1 - 0) - \gamma(x_1 - \delta) < e/4M,$$
$$\gamma(x_2 - 0) - \gamma(x_2 - \delta) < e/4M.$$

Then $\alpha_n(x_2 - 0)$, $\alpha_n(x_1 - 0)$, $\alpha(x_2 - 0)$, $\alpha(x_1 - 0)$ all differ from $\alpha_n(x_2 - \delta)$, $\alpha_n(x_1 - \delta)$, $\alpha(x_2 - \delta)$, $\alpha(x_1 - \delta)$ by less than $\frac{1}{4}e$ which is independent of n. But

$$\lim \left[\alpha_n(x_2 - \delta) - \alpha_n(x_1 - \delta)\right] = \alpha(x_2 - \delta) - \alpha(x_1 - \delta)$$

so that $\lim \left[\alpha_n(x_2 - 0) - \alpha_n(x_1 - 0)\right]$ differs from $\alpha(x_2 - 0) - \alpha(x_1 - 0)$ by less than e.

This theorem may also be compared with a corresponding theorem of Bray's (loc. cit.) on the continuity of

$$\int_c^d f(s) d_s \alpha(x, s).$$

Other theorems such as 1(3), 2(3), 2(4), 2(5) can also be applied to generalized Stieltjes integrals. In all cases we choose $h \equiv 1$ as the basic summable function and T_0 as the class of step-functions.

The frequent appearance of the function $h \geq 0$ summable I may seem unnecessary to some readers. In the usual applications of the general theory this h can be chosen equal to 1 and then it drops out of sight. But cases can be invented where this function would not be summable.

For example, let the elements p be the numbers $x (0 \leq x \leq 1)$, and let T_0 be the class of functions which are zero except at a finite number of values of x, that is of the type

$$f(x) = f_i \quad x = x_i \quad i = 1, 2, \cdots n,$$
$$= 0 \quad \text{otherwise.}$$

Let $s(x)$ be some positive function finite for each x and define

$$I(f) = \sum_{i=1}^{n} f_i s(x_i).$$

This integral satisfies all the required conditions and at any particular x it is possible to find many members of T_0 which do not vanish at that point. Nevertheless there is no summable function $h \geqq 0$ which is greater than zero on a set having the power of the continuum. For then $sh > 0$ everywhere on this set, which is the limit of the set $sh > e$ as $e \doteq 0$ $(e > 0)$. The limit of a non-contracting sequence of sets which are of countable power is also of countable power so that a number $e > 0$ would exist such that $sh > e$ on a set of power greater than the countable. But then the integral would be greater than e added to itself a countable number of times which is infinite.

If difficulties arise even with so simple an example, it can be understood why in the general case there is frequent reference to a basic summable $h \geqq 0$.

Note. The author wishes to apologize for a statement in the paper of which this is a continuation. On page 279 it was suggested that Moore's use of relatively convergent sequences was a restriction and on page 281 it was stated that the Moore integral (which is again referred to here in 3(2)) was a special instance of the author's. The latter statement is untrue and the former misleading. In a sense there is a restriction because the class of functions to which Moore's integral can be applied is given initially and is not progressively extended to so wide a class, but on the other hand the restrictions placed on the integral operation of Moore are not so stringent. As in 3(5) the sum of a convergent series is a form of Moore integral. Our theory would consider only absolutely convergent series in the corresponding case.

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SUMMABILITY OF DOUBLE SERIES.

By Lloyd L. Smail.

Practically nothing has as yet been done in the application of methods of summation of divergent series to double series. C. N. Moore* has discussed certain aspects of the Cesàro summability of double series, with the object in view of applications to double Fourier's series. Bromwich and Hardy† have given an extension of Abel's theorem on the continuity of power series to double series summable by an extension of Hölder's method.

The object of this paper is to give a general theorem on a method of summation of double series analogous to the general method of summation for simple series which I gave in my paper on "A General Method of Summation of Divergent Series" in the Annals of Mathematics for December, 1918.

Let $f_{i,j}(m, n, x, y)$ be a function defined for all positive integral and zero values of i and j, and for all real values of m, n, x, y, and satisfying the following conditions:

1°. When m, n, x, y are fixed, for every i, j,

(*a*) $$f_{i,j} > 0,$$

(*b*) $$h_{i,j} \equiv f_{i,j} - f_{i,j+1} > 0,$$

(*c*) $$h'_{i,j} \equiv f_{i,j} - f_{i+1,j} > 0,$$

(*d*) $$H_{i,j} \equiv f_{i,j} - f_{i+1,j} - f_{i,j+1} + f_{i+1,j+1} > 0;$$

2°. $\underset{x,y}{\mathbf{L}}\ \underset{m,n}{\mathbf{L}} f_{i,j}(m, n, x, y) = 1$* for i, j fixed;

* By this notation $\underset{x,y}{\mathbf{L}}\ \underset{m,n}{\mathbf{L}}(\ )$, we mean $\underset{x,y}{\mathbf{L}} \{ \underset{m,n}{\mathbf{L}}(\ ) \}$.

3°. $\underset{x,y}{\mathbf{L}}\ \underset{m,n}{\mathbf{L}} f_{i,n}(m, n, x, y) = 0$ for i, j fixed;

4°. $\underset{x,y}{\mathbf{L}}\ \underset{m,n}{\mathbf{L}} f_{m,j}(m, n, x, y) = 0$ for i, j fixed;

5°. $\underset{x,y}{\mathbf{L}}\ \underset{m,n}{\mathbf{L}} f_{m,n}(m, n, x, y) = 0.$

* C. N. Moore, "On Convergence Factors in Double Series and the Double Fourier's Series," Trans. Am. Math. Soc., vol. XIV, p. 73.

C. N. Moore, "On the Summability of the Double Fourier's Series of Discontinuous Functions," Math. Ann., vol. LXXIV, p. 555.

† Bromwich and Hardy, "Some Extensions to Multiple Series of Abel's Theorem on the Continuity of Power Series," Proc. London Math. Soc., vol. II, p. 161.

Let $\sum_{m,n} a_{m,n}$ $(m, n = 0, 1, 2, \cdots)$ be any given double series. If we form the expression $\sum_{i=0}^{m}\sum_{j=0}^{n} a_{i,j}f_{i,j}$, and the limit

$$\text{(I)} \qquad \mathop{\mathrm{L}}_{x,y}\mathop{\mathrm{L}}_{m,n}\sum_{i=0}^{m}\sum_{j=0}^{n} a_{i,j}f_{i,j}(m, n, x, y) = S$$

exists, with a finite value S, we shall say that the double series $\sum_{m,n} a_{m,n}$ is *summable* by the summation-function $f_{i,j}(m, n, x, y)$, and that S is its *Sum* (or generalized sum).

One of the first problems in a general study of such a method of summation is to determine whether a convergent series is always summable by the method.

As our definition of convergency of a double series we shall take the following: If $S_{m,n} = \sum_{i=0}^{m}\sum_{j=0}^{n} a_{i,j}$ approaches a definite limit S as m and n increase indefinitely, simultaneously but independently, and if for every m, n, we have

$$(1) \qquad |S_{m,n}| < C \text{ (a positive constant)},$$

then $\sum_{m,n} a_{m,n}$ is convergent with sum S.*

THEOREM. *If the double series* $\sum_{m,n} a_{m,n}$ *is convergent with sum* S, *according to the above definition, then the series will also be summable by definition* (*I*), *with the summation-function* $f_{i,j}$, *with generalized sum* S, *provided* $f_{i,j}$ *satisfies the conditions* 1°–5° *above.*

For since $\sum_{i,j} a_{i,j}$ is convergent, we may write

$$(2) \qquad S_{i,j} = S + \delta_{i,j}, \qquad |\delta_{i,j}| < \epsilon, \qquad i, j \geqq M,$$

where ϵ is any arbitrarily small positive number.

Then by the extension of Abel's identity or partial summation formula for double series,† we have

$$\sum_{i=0}^{m}\sum_{j=0}^{n} a_{i,j}f_{i,j} = \sum_{i=0}^{m-1}\sum_{j=0}^{n-1} S_{i,j}H_{i,j} + \sum_{j=0}^{n-1} S_{m,j}h_{m,j} + \sum_{i=0}^{m-1} S_{i,n}h'_{i,n} + S_{m,n}f_{m,n}.\text{‡}$$

Taking $m, n > M$, and for $i, j \geqq M$ substituting $S_{i,j} = S + \delta_{i,j}$, we get

* Some writers omit the condition (1) in their definition of convergence, and regard this condition, called the "condition of finitude," as a restriction on convergency. See references to Bromwich and Hardy, and Moore above; also Bromwich, "Infinite Series," §§ 29, 37.

† Hardy, Proc. London Math. Soc., Vol. I, p. 124.

‡ Condition 1° is required for the application of Abel's identity.

$$\sum_{i=0}^{m}\sum_{j=0}^{n} a_{i,j}f_{i,j} = \left(\sum_{i=0}^{M-1}\sum_{j=0}^{M-1} S_{i,j}H_{i,j} + \sum_{i=0}^{M-1}\sum_{j=M}^{n-1} S_{i,j}H_{i,j} + \sum_{i=M}^{m-1}\sum_{j=0}^{M-1} S_{i,j}H_{i,j}\right.$$

$$(3) \qquad \left. + \sum_{j=0}^{M-1} S_{m,j}h_{m,j} + \sum_{i=0}^{M-1} S_{i,n}h'_{i,n}\right) + S\left(\sum_{i=M}^{m-1}\sum_{j=M}^{n-1} H_{i,j} + \sum_{j=M}^{n-1} h_{m,j} + \sum_{i=M}^{m-1} h'_{i,n}\right)$$

$$+ \left(\sum_{i=M}^{m-1}\sum_{j=M}^{n-1} \delta_{i,j}H_{i,j} + \sum_{j=M}^{n-1} \delta_{m,j}h_{m,j} + \sum_{i=M}^{m-1} \delta_{i,n}h'_{i,n}\right) + S_{m,n}f_{m,n}.$$

If we keep M fixed, and take the double limit $\underset{x,y}{\mathbf{L}}\ \underset{m,n}{\mathbf{L}}\ (\ )$ on both sides of this equation, we find, by making use of conditions 2°–5° and relations (1) and (2), that

$$\underset{x,y}{\mathbf{L}}\ \underset{m,n}{\mathbf{L}} \sum_{i=0}^{m}\sum_{j=0}^{n} a_{i,j}f_{i,j}(m, n, x, y) = S.$$

Thus our general theorem is proved.

Now suppose that we take for $f_{i,j}$ the special form

$$(4) \qquad f_{i,j}(m, n, x, y) = f_i(m, x)\cdot f_j(n, y),$$

where $f_i(m, x)$ satisfies the conditions:

(α) $f_i(m, x) > 0$ and $f_i - f_{i+1} > 0$ for every i, m;

(β) $\underset{x}{\mathbf{L}}\ \underset{m}{\mathbf{L}}\, f_i(m, x) = 1$ for i fixed;

(γ) $\underset{x}{\mathbf{L}}\ \underset{m}{\mathbf{L}}\, f_m(m, x) = 0$.

Then it can be shown without any difficulty that this function $f_{i,j}(m, n, x, y) = f_i(m, x)\cdot f_j(n, y)$ satisfies the conditions 1°–5° preceding.

The familiar methods of Cesàro, Hölder, Borel, LeRoy, Riesz, de la Vallée-Poussin, Plancherel, etc., all satisfy the conditions (α), (β), (γ),* so that the summation-function of any one of these well-known methods can be used to build up the summation-function for the summability of double series. If we take the Cesàro function, our general theorem gives as a special case the result of Moore (loc. cit., p. 81).

UNIVERSITY OF WASHINGTON,
March 18, 1919.

* See Annals of Mathematics, Dec., 1918, p. 154; also my Columbia Dissertation, 1913.

THE FUNDAMENTAL THEOREM OF CELESTIAL MECHANICS.

By J. L. Coolidge.

We mean by this sonorous title the classical theorem which tells us that the center of gravity of a planet traces with regard to the sun, a conic having one focus at the sun's center. The following proof is offered in the hope that it may be found simpler than some of those in current use.

We start with the Newtonian law of universal gravitation and give the usual proof that since we have a central force the point must trace a plane curve. The plane of this curve we take as the z plane, place the origin at the center of the sun, and write the familiar equations of motion

$$x'' = \frac{kx}{r^3}, \qquad y'' = \frac{ky}{r^3}, \tag{1}$$

$$xy'' - yx'' = 0,$$

$$xy' - yx' = C.$$

At this point, instead of seeking immediately for the curve we want, let us rather look for its polar reciprocal in a unit circle with center at the origin. The equation of a tangent to our original curve is

$$Xy' - Yx' = C.$$

Its pole in the unit circle is

$$X = \frac{y'}{C} \qquad Y = -\frac{x'}{C}.$$

We seek the curvature of this polar curve

$$X' = \frac{y''}{C} = \frac{ky}{Cr^3}, \qquad Y' = -\frac{x''}{C} = \frac{-kx}{Cr^3},$$

$$X'' = \frac{k}{C}\frac{(ry' - 3yr')}{r^4}, \qquad Y'' = \frac{-k}{C}\frac{(rx' - 3xr')}{r^4},$$

$$\frac{X'Y'' - Y'X''}{(X'^2 + Y'^2)^{3/2}} = \frac{k^2}{C^2}\frac{(xy' - yx')}{r^6} \div \frac{k^3}{C^3}\left(\frac{x^2 + y^2}{r^6}\right)^{3/2} = \frac{C^2}{k}.$$

Since this polar curve has a constant curvature it is a circle.
Hence the original curve was a conic with a focus at the origin.

American Expeditionary Forces,
October, 1918.

ALGEBRAIC SURFACES, THEIR CYCLES AND INTEGRALS.

By S. Lefschetz.

CONTENTS.

Introduction.

The theory to be considered here has been created by Emile Picard.* There are two important Mémoires due to Poincaré bearing largely on the topological phases of it.† A short but very interesting note by J. W. Alexander‡ and one by the writer§ practically make up the bibliography of the question.

It seems a reasonable requirement that purely topological properties should be derived by direct topological methods, and this requirement we have endeavored to meet here. In this we have proceeded along the

* Its exposition forms the major part of the Picard-Simart Traité des fonctions algébriques de deux variables. Paris, Gauthiers-Villars, 2 vols.

† Sur les cycles des surfaces algébriques, Jour. de Maths., ser. 5, vol. 8 (1902), pp. 169–214. Sur les périodes des intégrales doubles, ibid., ser. 6, vol. 2 (1906), pp. 135–189. These papers will be called in the sequel, "first Mémoire" and "second Mémoire."

‡ Sur les cycles des surfaces algébriques . . . , Rendiconti dei Lincei, ser. 5, vol. 23 (1914), pp. 55–62.

§ Sur certains cycles à deux dimensions des surfaces algébriques, Rendiconti dei Lincei, ser. 5, vol. 26 (1917), pp. 228–234.

lines of Poincaré's second Mémoire simplifying his proofs and completing his results at many important points. This has been followed by the transcendental theory wherein Picard's work on double integrals of the second kind has been much simplified: (*a*) By making their reduction to a simpler form rest upon a preliminary treatment of periodicity. (*b*) By developing their theory without making use of integrals of total differentials of the third kind. In fact reversing Picard's order we have shown that the properties of these integrals and of his number d may be deduced from those of double integrals. The paper concludes with a summary of Severi's related theory of the base and a rapid mention of some allied recent developments, in part unpublished, due to the writer.

Preliminary notions.

§ 1. *Manifolds and their cycles.**

1. Given a system of real equations with real variables

$$x_i = \theta_i(u_1, u_2, \cdots, u_n); \qquad (i = 1, 2, \cdots, n' > n),$$

where in the range of variation of the (u)'s the (θ)'s are continuous, analytical, and with functional determinants nowhere all zero, we may obtain by analytical continuation a new system and so on. To some of these systems may be adjoined certain conditional inequalities $\varphi(u_1, u_2, \cdots, u_n) > 0$. The point set thus defined constitutes an n-dimensional *manifold* M_n in n-space. By changing one of the inequalities into an equation we obtain a *boundary* of M_n and it will be assumed that the boundaries are grouped into $(n-1)$-dimensional manifolds $M^1{}_{n-1}, M^2{}_{n-1}, \cdots, M^s{}_{n-1}$.

The familiar notion of the two directions assignable to a line segment may be extended to a certain class of manifolds, those said to be "two-sided." If a small directed circuit drawn on an M_2 can be displaced, then returned to its first position with changed sense, M_2 is said to be *one-sided*. If the sense is invariant M_2 is said to be *two-sided*, and each sense may be considered as defining a *sense* of M_2, so that when we invert the circuit we agree to consider M_2 as replaced by $-M_2$.

When M_2 is thus sensed its boundaries will also be if we make the convention that a bounding line is to be so sensed that when the above directed circuit comes into contact with it there is agreement in the senses impressed by circuit and bounding line upon the small arc common

* For a more complete discussion along the same line see Poincaré, Jour. de l'Ec. Polyt., ser. 2, vol. 1 (1895), § 3; Picard-Simart, vol. 1, p. 20; Heegard, Bull. Soc. de Math., vol. 44 (1916), p. 197. For a more formal treatment better adapted to questions of pure analysis situs see the Dehn-Heegard article on the subject in the Enzyk. der math. Wiss., III, AB3, also a paper by Veblen and Alexander in the Annals of Math., ser. 2, vol. 14 (1913), p. 163.

to them. All this can be readily extended to an M_n. *Only two-sided manifolds will be considered here.*

The relation between M_n and its sensed boundaries will be expressed by a congruence $M_n \equiv \Sigma M^i_{n-1}$, or $M_n \equiv 0$ when M_n is *closed* (without boundaries). If we merely wish to indicate that the (M_{n-1})'s bound some M_n in an $M_{n'}$ we shall write a *homology* $\Sigma M^i_{n-1} \sim 0 \pmod{M_{n'}}$, or simply $\Sigma M^i_{n-1} \sim 0$, when no doubt exists as to the $M_{n'}$ in question.

The term "manifold" will be extended to cover an algebraic sum of manifolds as just defined, the boundary being then defined as the algebraic sum of the boundaries. When the (M_n)'s are in an $M_{n'}$, $\Sigma\lambda_i M_n{}^i$, (λ_i integer) may be interpreted as an aggregate of λ_1 manifolds very near $M_n{}'$ in $M_{n'}$, of λ_2 manifolds very near $M_n{}^2$ in $M_{n'}$, etc., the manifolds being all united into one by n-dimensional "tubes" (pairs of opposite segments if $n = 1$, ordinary tubes if $n = 2$, etc.). With these definitions we are enabled to combine congruences and homologies by addition after multiplying them by arbitrary integers. A manifold in the extended sense may now be "connected" without necessarily behaving in the neighborhood of every point not on the boundary like the interior of an n-dimensional sphere.

A closed M_k (of the extended type) contained in an M_n (of the restricted type) is called a *k-dimensional cycle* of M_n (linear cycle if $k = 1$, zero-cycle if $M_k \sim 0 \pmod{M_n}$).

Cycles are independent if they satisfy no homology. The number R_k of independent k-dimensional cycles of M_n is called its kth *index of connectivity* (linear index if $k = 1$). The numbers $1 + R_k$ have also been called Betti numbers by Poincaré who proved most of their known properties and in particular showed that $R_k = R_{n-k}$.*

We have already mentioned displacements. Only such displacements of an M_n will be considered during which it generates an M_{n+1}. If M_n is displaced in an $M_{n'}$, its extreme positions are equivalent mod. $M_{n'}$, for they form the boundary of the manifold generated during the motion.

Remark.—It has been tacitly assumed that all manifolds considered are finite. We may introduce infinite manifolds by means of systems,

$$\frac{1}{x_i} = \theta_i(u_1, u_2, \cdots, u_n),$$

where the (θ)'s have the same properties as previously.

2. On a closed sensed M_2 of the restricted type we consider the two linear cycles α, β, concerning which we assume that they have only a finite number of intersections simple for each cycle and at which they are

* Complément à l'Analyse Situs, Rendiconti del Circ. Mat. di Palermo, vol. 13 (1899).

not tangent. If these conditions are not fulfilled we deform one of the cycles until they are. From one of the points of intersection O we take a small segment OA on α in its positive direction and a similar segment OB on β and we join them by a small arc AB of M_2 not cut in two by one of the cycles. Finally we sense OAB so that its vertices are met in the order named. This being done for all intersections there will be some, n in number, for which the triangle is a positive circuit of M_2 and others, n' in number, for which it is just the opposite. The difference $n - n'$ will be designated by (α, β). This character was first introduced by Poincaré,* who proved its properties and made extensive use of it in his investigations. Among these properties the following are easily verified.

(a) $(\alpha, \beta) = -(-\alpha, \beta) = -(\beta, \alpha)$.

(b) $(\alpha + \beta, \gamma) = (\alpha, \gamma) + (\beta, \gamma)$.

(c) If α or β are ~ 0, then $(\alpha, \beta) = 0$.

A similar character to be designated by (M_p, M_{n-p}) may at once be established for two cycles M_p, M_{n-p} of a closed M_n of the restricted type, (a) and (b) following then immediately. It is then easy to show that (d) if M_q, $q > p$ is closed of the restricted type and goes through M_p intersecting M_{n-p} in an M_{q-p} then (M_p, M_{q-p}) computed relatively to M_q is equal to $\pm (M_p, M_{n-p})$ computed relatively to M_n. By means of (d) we can then extend (c). Properties (b), (c) show that (M_p, M_{n-p}) is really a simultaneous invariant of the two cycles with respect to the operation of "homology."

If $p = n - 1$ we have in addition this: When M_{n-1} is not ~ 0, there is a cycle M_1 such that $(M_p, M_1) = +1$. It is the cycle known to exist which meets M_p at one point only, or else its opposite. Hence M_{n-1} is not a zero cycle if and only if there is a linear cycle M_1 such that $(M_n, M_1) \neq 0$.

3. We shall have occasion to consider simple and double integrals. Nothing need be said about the former.

By

$$\sum_{j,k} \int\int_{M_2} R_{jk}(x_1, x_2, \cdots, x_n)dx_j dx_k, \qquad (M_2 \text{ finite})$$

is meant this: M_2 can be subdivided into a finite number of domains M_2^i within which the (x)'s are analytical in two variables u_1, u_2, their jacobians being nowhere all zero, and it is assumed that the (R)'s are analytical in the same conditions. Denoting by M_2^i the image of M_2^i in the u_1u_2 plane the double integral is by definition equal to

* Jour. Ec. Polyt., § 9. He uses the notation $N(\alpha, \beta)$.

$$\sum_{i,j,k} \int\int_{\overline{M}_2^i} R_{jk} \frac{D(x_j, x_k)}{D(u_1, u_2)} du_1 du_2.$$

The double integrals in the sum can be reduced to line integrals extended over the boundaries of the $(\overline{M}_2)$'s and the directions in which these line integrals must be taken on the boundaries are determined by the order in which u_1 and u_2 are taken. Thus this order determines a direction on the boundaries hence also on those of the manifolds M_2^i and finally a sense on M_2. Conversely if we assign a sense on M_2, we determine the order in which u_1 and u_2 must be taken and therefore the sign of the integral. Thus there is a definite meaning to be attached to an integral over a sensed M_2. From the manifold unsensed we may obtain the absolute value of the integral and once the sense is known we may also obtain the sign. This is obviously similar to the case of simple integrals and may be extended to integrals of any multiplicity.

§ 2. *Algebraic curves and their integrals.*

4. A plane algebraic curve C of order m, is the locus of the points whose real or complex coördinates satisfy an equation $f(x, y) = 0$, where $f(x, y)$ is a polynomial of order m. The curve is said to be irreducible if f itself is. The function $y(x)$ thus defined is called an *algebraic function* of x.

In general C has multiple points but it is always reducible to a curve having only double points with distinct tangents for singularities. This reduction can be effected by a *birational transformation* that is, by a transformation such that the coördinates of a point on one curve are rational in those of the corresponding point on the other and vice versa. We shall then assume that C has only these simple singularities and moreover that it lies in general position with respect to the axes and the line at infinity.

A Riemann surface is defined as an M_2 whose points are in point to point correspondence without exception with the real and complex points of C. We may obtain one thus: Let

$$x = x' + ix'', \qquad y = y' + iy'',$$

$$f(x, y) = P(x', x'', y', y'') + iQ(x', x'', y', y''),$$

where the new quantities introduced are all real. The M_2 represented in the real four-space of the variables x', x'', y', y'' by $P = Q = 0$ answers the question.

The Riemann surface is a closed two-sided manifold. It is obviously closed and that it is two-sided may be shown thus: The infinite and

branch points are ordinary points, hence a displacement of a sensed circuit ζ returning it to its original position with sense inverted can be assumed such that ζ never crosses these points. Were such a displacement possible it would also be possible to invert by a finite displacement the circuit ζ' projection of ζ in the plane $y' = y'' = 0$ (complex x plane) which is not the case.

The linear index R_1 of the Riemann surface is even and given by

$$R_1 = (m-1)(m-2) - 2\delta = 2p,$$

where δ is the member of double points and p is the *genus*. Both R_1 and p are invariants under birational transformations.

There are $2p + m - 1$ linear cycles of which no combination bounds a finite part of the surface.

5. By *abelian integral* belonging to C is meant one of the type

$$(1) \qquad \int R(x, y)dx,$$

where $R(x, y)$ is a rational point-function on C. This integral can have no other singularities than poles or logarithmic points in finite number. The integral is said to be of the *first kind* if it is finite everywhere and it is said to be of the *second* or *third* kind according as it has or has not poles for only singularities.

The value of (1) taken over a path with fixed extremities is unchanged when the path does not cross a logarithmic point. Any other path with the same extremities leads to a value differing from the first by a *period*, that is by the value of (1) taken over a linear cycle. The period is *logarithmic* if the cycle is merely a circuit surrounding a logarithmic point, *cyclic* if otherwise. *The sum of the logarithmic periods is equal to zero.*

An abelian integral (1) without logarithmic points at finite distance can be reduced by subtraction of a rational fraction $S(x, y)$ to the type

$$(2) \qquad \int \frac{Q(x, y)dx}{f_y'},$$

where Q is a polynomial *adjoint* to C, that is vanishing at its double points. $S(x, y)$ may be determined by operations rational with respect to the coefficients of R and f.

The integral (2) has $2p + m - 1$ distinct periods and by properly choosing Q these may be made to assume arbitrary values. They correspond to as many cycles of which no combination bounds a finite part of the Riemann surface.

If (1) is without periods it is equal to a rational fraction which may be rationally determined in the same sense as above.

Topology of algebraic surfaces.

§ 1. *Generalities.*

6. The definition of algebraic surfaces is the same as that given for algebraic curves. We shall consider throughout a surface F of order m, equation $F(x, y, z) = 0$, and we assume it to be irreducible, in general position with respect to the axes and the plane at infinity and to have no other singularities than a double curve, a finite number of triple points triple also for the curve, and of pinch-points, or points of the double curve where the two tangent planes coincide. It may be proved that any algebraic surface is birationally transformable into one having no other singularities than these.*

An algebraic surface intersects F in a finite number of space algebraic curves or curves of which the projections on any plane are plane algebraic curves. A linear system of surfaces cuts out in F a linear system of curves and if C is the generic curve of the system, the latter is usually denoted by $|C|$. The system of the plane sections will be denoted by $|H|$ and the pencils cut out by the planes $x = \text{Ct.}$, $y = \text{Ct.}$, will be denoted by $\{H_x\}$, $\{H_y\}$.

It may be stated here that with some slight changes in wording $|H|$ could be everywhere replaced by a linear system $|E|$ of dimensionality $s \geqq 2$, not containing a subsystem of ∞^{s-2} reducible curves,† and such that there is no simple point of F multiple for all the curves E through it.

The surface F may be represented by an M_4 belonging to a real six-space, and often called its Riemann image. Within this M_4 the algebraic curves of F are represented by (M_2)'s and in the sequel no distinction will be made between curves and surfaces and their representative manifolds. That the M_4 is two-sided may be proved as for Riemann surfaces.

§ 2. *Linear and three-dimensional cycles.*

7. An arbitrary linear cycle Γ_1 of F may be composed of several distinct circuits, but if so we may join them by pairs of opposite segments so as to obtain a cycle $\sim \Gamma_1$ and composed of a single circuit. Let us suppose then that Γ_1, already possesses this property.

Let A be a base point of $\{H_x\}$. We may add to Γ_1 a pair of opposite segments going from A to a point on the cycle, which, still possessing the above property, will now go through A. Let B be an arbitrary point of the cycle and in its H_x join A to B by an arc AB. As B starts from A and describes Γ_1, AB first reduced to the point A varies continuously in

* Beppo Levi, Annali di mat., ser. 2, vol. 26.

† The necessity of this restriction is explained in the course of the proof of a certain fundamental theorem (No. 11).

F and when B returns to A it reduces to a closed circuit Γ_1', contained in a certain curve H_{x_0}. $\therefore \Gamma_1 \sim \Gamma_1'$ since they bound the M_2 swept out by the arc AB. For a similar reason if we displace H_{x_0} from its position to that of any other H, Γ_1' will take in H a position $\Gamma_1'' \sim \Gamma_1'$, hence $\Gamma_1'' \sim \Gamma_1' \sim \Gamma_1$ and finally we see that *any linear cycle is homologous to one contained in an arbitrary plane section.**

Remark. Obviously this proof holds if $\{H_x\}$ be replaced by any linear pencil having at least *one* base point (pencil of positive degree). Any linear cycle is homologous to one contained in an arbitrary curve of the pencil.

8. Let us draw cross cuts from one point y_0 of the y plane to the critical points $a_1, a_2, \cdots, a_N$, or points for which the plane $y =$ Ct., is tangent to the surface whose class is therefore N. It is only when y is a critical value that H_y may lose cycles for then and then only its genus decreases. Hence a cycle which is returned to a homologous position when y turns around any critical point is also returned to such a position when y describes any closed circuit, and for this reason will be called *invariant.* When y describes its plane without crossing the cuts, an invariant cycle Γ_1 generates a three-dimensional manifold Γ_3 bounded by the locus of Γ_1, when y describes the lips of the cuts. But owing to the invariance of Γ_1, this boundary is composed of a sum of mutually opposite parts, hence Γ_3 is a three-dimensional cycle. Conversely such a cycle Γ_3 intersects H_y into an invariant linear cycle Γ_1.

If Γ_3 forms boundary on F so does Γ_1 on H_y (and a fortiori on F) and conversely if Γ_1 forms boundary on F, such is also the case for Γ_3. For first if Γ_1 bounds an M_2 as y describes its plane this M_2 will sweep an M_4 bounded by Γ_3. Next let Γ_1' be any linear cycle of F, Γ_1'' the homologous cycle in H_y. We have $(\Gamma_3, \Gamma_1') = (\Gamma_3, \Gamma_1'')$, $(\Gamma_3, \Gamma_1'') = \pm (\Gamma_1, \Gamma_1'')$, (No. 2). Now if $\Gamma_3 \sim 0$, $(\Gamma_3, \Gamma_1') = 0$ whatever Γ_1' and therefore also $(\Gamma_1, \Gamma_1'') = 0$, and as Γ_1'' is after all a perfectly arbitrary cycle of H_y, $\Gamma_1 \sim 0 \bmod H_y$ and a fortiori mod F.

Remark. We have not shown that any non-zero cycle Γ_1 of F contained in an H_y is the trace of a Γ_3, but only that this is the case for invariant cycles. In point of fact it is shown below that only a certain multiple of Γ_1 is homologous to a trace of Γ_3, mod F.

9. Any H_y contains a cycle Γ_1 homologous to an arbitrary one of F and composed of a single circuit at all of whose points the arc is analytical. Γ_1 may have multiple points but we may assume that the branches through them have distinct tangents as this condition can always be fulfilled by slightly deforming Γ_1. The curves of $\{H_x\}$ which meet Γ_1

* Compare with Picard-Simart, vol. I, p. 85, also Poincaré, 1st Mémoire, § 4, p. 200.

determine a three-dimensional cycle Γ_3 behaving everywhere like an M_3 in the restricted sense except perhaps at the base points of the pencil and also along a certain multiple manifold. However the various branches of Γ_3 through an arbitrary point of this manifold have again the indicated behavior. H_y intersects Γ_3 into several circuits besides Γ_1, say Γ_1', Γ_1'', $\cdots$, $\Gamma_1^{(n-1)}$, with $n \leqq m$, for on each circuit must lie at least one of the m intersections of H_y with an H_x of Γ_3. As y varies the circuits vary also, but two of them $\Gamma_1^{(i)}$, $\Gamma_1^{(j)}$ *can never coalesce into one.* For were this to happen for $y = y_0$ as y would tend towards y_0 the complete intersection of H_y and Γ_3 would acquire a multiple point A which would disappear when y would leave y_0 along at least one path through it. But the H_x through A must then be part of the multiple manifold of Γ_3 and therefore there must be on the intersection in question a multiple point tending towards A when y approaches y_0 along any path whatever.

Now since Γ_3 is a connected manifold we can always make y describe a closed circuit so chosen that $\Gamma_1^{(i)}$ is returned to H_y in the position of $\Gamma_1^{(j)}$ hence $\Gamma_1^{(i)} \sim \Gamma_1^{(j)}$, that is Γ_1, Γ_1', $\cdots$, $\Gamma_1^{(n-1)}$ are homologous cycles of F. It follows that Γ_3 intersects H_y into a linear cycle obviously invariant, $\sim n\Gamma_1$. *Hence some multiple of any cycle of F is homologous to an invariant cycle, mod. F.*

It will be observed again that if Γ_1 is in H_y, $n\Gamma_1$ may be homologous to an invariant cycle, mod. F, though not mod. H_y.

It follows from the above that the number of independent invariant cycles, is the same as the number R_1 of independent linear cycles of F. But from what we have seen in No. 7 follows that the number of independent invariant cycles is equal to the number R_3 of independent three-dimensional cycles. Hence $R_1 = R_3$, which is Poincaré's theorem for the special case of algebraic surfaces.

§ 3. *Proof that the index R_1 is even. Irregularity of the Surface.*

10. The property to be derived presently is usually obtained by analytical methods.* Our treatment here will be wholly topological.

Let us draw on H_y as many independent invariant cycles as we may, say Γ_1^1, Γ_1^2, $\cdots$, Γ_1^q such that $(\Gamma_1^i, \Gamma_1^j) = 0$, $(i \neq j,\ i, j = 1, 2, \cdots, q)$. Since they are independent cycles of H_y we may find q other cycles of it, say Γ_1^{q+1}, Γ_1^{q+2}, $\cdots$, Γ_1^{2q}, such that Γ_1^{q+1} meets Γ_1^i, but not Γ_1^j, $i \neq j < q$. Assuming these new cycles properly sensed we will have

$$(\Gamma_1^i, \Gamma_1^{q+1}) = +1, \qquad (\Gamma_1^i, \Gamma_1^{q+j}) = 0, \qquad i \neq j.$$

I say now *that the $2q$ cycles Γ_1^i, Γ_1^{q+i} are independent cycles of F.* For assume a relation

* See for example Picard-Simart, vol. II, p. 423. The linear index is denoted there by r.

$$\sum_{i=1}^{q} \lambda_i \Gamma_1{}^i + \sum_{i=1}^{q} \mu_i \Gamma_1{}^{q+i} \sim 0$$

and let $\Gamma_3{}^i$, $i \leqq q$, be the three-dimensional cycle of which the invariant cycle $\Gamma_1{}^i$, is the trace on H_y. We have

$$\sum \lambda_i (\Gamma_3{}^j, \Gamma_1{}^i) + \sum \mu_i (\Gamma_3{}^j, \Gamma_1{}^{q+i})$$
$$= \sum \lambda_i (\Gamma_1{}^j, \Gamma_1{}^i) + \sum \mu_i (\Gamma_1{}^j, \Gamma_1{}^{q+i}) = \mu_j = 0 \qquad (j = 1, 2, \cdots, q);$$
$$\therefore \sum \lambda_i \Gamma_1{}^i \sim 0, \qquad (\text{mod. } F),$$

and since the cycles at the right are independent mod. F, it must be that the (λ)'s are all zero which proves our statement.

Consider now any linear cycle Γ_1 of F assumed in H_y and let

$$(\Gamma_1{}^i, \Gamma_1) = \nu_i, \qquad i \leqq q.$$

The cycle $\bar{\Gamma}_1 = \Gamma_1 - \Sigma \nu_i \Gamma_1{}^{q+i}$ will satisfy the relations $(\Gamma_1{}^i, \bar{\Gamma}_1) = 0$, $i \leqq q$. By making y describe properly chosen closed paths, $\bar{\Gamma}_1$ will be returned to certain cycles $\bar{\Gamma}_1, \bar{\Gamma}_1{}^2, \cdots, \bar{\Gamma}_1{}^{(n-1)}$ all $\sim \bar{\Gamma}_1$, mod. F and such that

$$\bar{\bar{\Gamma}}_1 = \bar{\Gamma}_1 + \bar{\Gamma}_1{}^1 + \cdots + \bar{\Gamma}_1{}^{n-1}$$

is an invariant cycle. We have

$$(\Gamma_1{}^i, \bar{\Gamma}_1{}^j) = (\Gamma_1{}^i, \bar{\Gamma}_1) = 0, \qquad \therefore (\Gamma_1{}^i, \bar{\bar{\Gamma}}_1) = 0, \qquad (i \leqq q).$$

It follows from this that $\bar{\bar{\Gamma}}_1$ must be dependent upon the cycles $\Gamma_1{}^i$, $i \leqq q$ for otherwise there would be $q + 1$ independent invariant cycles satisfying the relations

$$(\Gamma_1{}^i, \Gamma_1{}^j) = 0, \qquad (i, j \leqq q + 1),$$

while we assumed that there were at most q. Hence finally *all linear cycles of F are dependent upon the cycles* $\Gamma_1{}^i$, $\Gamma_1{}^{q+i}$, $i \leqq q$, and as these are independent it follows that $R_1 = 2q$ *that is R_1 is even* as was to be proved.

The integer q is called the *irregularity* of F. It is its most important invariant with respect to birational transformations.

§ 4. *Linear cycles of the plane sections.*

11. We propose to examine the mode of variation of the linear cycles of H_y as y varies. We have already stated (no. 7) that it suffices to examine what happens near the critical points. Consider for the moment H_y as the classical m-sheeted Riemann-surface image of $z(x)$ when y is fixed, and let $\alpha_1, \alpha_2, \cdots, \alpha_n$ be its branch points. When y tends towards a_i two of these uniting the same two sheets, say α_1, α_2, tend to coincide

and the cycle δ_i surrounding the two points in one of the sheets is certainly invariant when y turns around a_i. Let A, B be two points in the Riemann surface belonging each to one of the sheets united in α_1, α_2. There are two possibilities accordingly as there is or is not a path going from A to B without turning around either α_1 or α_2 alone.

(*a*) In the first case the path in question followed by going from B to the vicinity of α_1, rotating around it once and returning to A in its sheet yields a cycle δ_i' other than δ_i. *This new cycle is increased by* δ_i *when* y *turns around* a_i. If α_2 had taken the place of α_1 we would have in place of δ_i' the cycle $\delta_i' - \delta_i$ of similar behavior, from which we can easily infer that any cycle when increased by a proper multiple of δ_i' is unchanged when y turns around a_i. Hence in the same conditions any cycle is increased by a multiple of δ_i.

(*b*) In the second case, for $y = a_i$, the critical points α_1, α_2 will merge into a double point and the two sheets of the Riemann surface will cease to be united there. For $y = a_i$ it will be impossible to go from A to B, that is the curve H_{a_i} is *reducible*. As the pencil $\{H_y\}$ is arbitrary in $|H|$ any similar pencil must contain reducible curves, hence there are ∞^2 reducible plane sections. According to Castelnuovo* F is then a *ruled* or a *Steiner surface* (surface of order four with three concurrent but not coplanar double straight lines). In case of a ruled surface the reducible sections are those by planes through the generators, while for the Steiner surface they are those by the tangent planes, the sections being then composed of two conics.

We can thus state the following fundamental theorem: *To each critical point* a_i *corresponds a cycle* δ_i *of* H_y *invariant in its vicinity, and such that when* y *turns around* a_i *any other cycle is increased by a multiple of it.*

Remark. This theorem holds when $|H|$ is replaced by a system $|E|$ such as described in No. 6. For the Steiner or ruled surface we may take for example $|E| = |2H|$.

§ 5. *Two-dimensional cycles.*†

12. An arbitrary two-dimensional cycle Γ meets an arbitrary algebraic curve and in particular H_y in a finite number of points only, and certainly this condition will be fulfilled if the cycle is adequately deformed.

When y describes a positive circuit around y_1, the points of intersection of the H_y and Γ describe small circuits which may be positively or negatively related to Γ. Let $\theta_1, \theta_2, \cdots, \theta_n$ be those of the first type, $\theta_1', \theta_2', \cdots, \theta_{n'}'$ those of the second. We have in fact $n - n' = (H_y, \Gamma)$, and we remark that:

* Rendiconti dei Lincei, ser. 5, vol. 3 (1894), pp. 22–26.

† See Poincaré's second Mémoire.

(*a*) If Γ is replaced by an irreducible algebraic curve C of order μ, $n = \mu$, $n' = 0$ as follows at once from the fact that we can permute any two of the circuits $\eta_1, \eta_2, \cdots, \eta_\mu$, now taking the place of the (θ)'s by making y_1 describe adequate closed paths in the y plane.

(*b*) As y_1 varies $n - n'$ is constant, for since $H_{y_1} \sim H_{y_2}$

$$(H_{y_1}, \Gamma) - (H_{y_2}, \Gamma) = (H_{y_1} - H_{y_2}, \Gamma) = 0.$$

In particular if there is a single H_y that does not meet Γ then $n = n'$.

Now suppose we have both a curve $C = H$ and a cycle Γ such that $n \neq n'$. Then, keeping to the notation used above, we join the points on η_h of C to the points on θ_j and θ_k' of Γ by lines in H_y. Moreover, we denote by Ψ_{hj}, Ψ_{hk}' the tubes which these lines describe when y turns around y_1, and by (H), (Γ) the parts of these manifolds exterior to the (θ)'s and (η)'s. We have

$$\Psi_{hj} \equiv \eta_h - \theta_j, \qquad \Psi_{hk}' \equiv \eta_h - \theta_k', \qquad (H) \equiv \Sigma\eta_h, \qquad (\Gamma) \equiv \Sigma\theta_j - \Sigma\theta_k'.$$

$$\therefore m(\Gamma) - (n - n')(H) + \Sigma\Psi_{hj} - \Sigma\Psi_{hk}' \equiv 0.$$

The manifold at the left is closed, meets nowhere H_y and is reducible by deformation to $m\Gamma - (n - n')H$ cycle to which it is therefore homologous. In the same manner we can arrive at a cycle $\sim m\Gamma - (n - n')H$ meeting nowhere a group of curves H_y.

This reduction applies as well when $n = n'$. But in that case we may join the points of H_y describing θ_j and θ_j' by a line in H_y and obtain a tube Ψ_j resting on θ_j, θ_j'. With the same notations as above then

$$\Psi_j \equiv \theta_j - \theta_j', \qquad (\Gamma) \equiv \Sigma(\theta_j - \theta_j').$$

$$\therefore (\Gamma) - \Sigma\Psi_j \equiv 0,$$

and we have here a cycle $\sim \Gamma$ meeting nowhere H_y. Hence if there is an H_y which Γ does not intersect the cycle may be so deformed as not to meet a given group of (H_y)'s. In particular a finite cycle is reducible by a finite deformation to a cycle which does not meet a given group of (H_y)'s. That the deformation may be finite follows from our proof.

§ 6. *Reduction of two-dimensional cycles to a special type.*

13. When y describes the semi-straight line

(1) $$\text{argt.}\ (y - a_0) = \text{Ct.},$$

H_y generates a manifold K_y. If there are no critical points between two points y_1, y_2 of (1), the manifolds H_{y_1}, H_{y_2} are reducible to each other by a deformation never introducing any new singularities. This deforma-

tion sets up between them a point-to-point correspondence with the points at infinity corresponding to themselves. The point P_2 of H_{y_2} corresponding to a point P_1 of H_{y_1} will be called its projection in H_{y_2} and the locus of P_1 during the deformation its projecting line.

In the sequel it will be found convenient to denote the curves H_y corresponding to the critical points a_i by H_i, and the curves corresponding to $y = a_0, \infty$ by H_0 and H_∞ respectively.

Let Γ be a cycle which does not meet H_0, H_∞, or any of the curves H_i. Any finite cycle can be reduced by finite deformation to one having this property. If we project the points of Γ into H_0 the projecting lines form a finite M_3 bounded by $-\Gamma_1$, by a finite portion (H_0) of H_0, and by a certain manifold P that may be described thus: The manifold K_{a_i} corresponding to the semi-straight line through the critical point a_i intersects Γ in two sets of linear cycles such that for all points of one $|y - a_0| < |a_i - a_0|$, while for all points of the other, $|y - a_0| > |a_i - a_0|$. Let γ_i be the cycles of the second set. Then, the projection on H_0 of a system of closed linear cycles in K_y which tends towards γ_i as K_y tends towards K_{a_i} approaches one of two distinct limiting positions γ_i and γ_i' according as argt. $(y - a_0)$ tends towards $a_i - a_0$ from above or below. The projecting lines in the two cases form manifolds with limiting positions P_i', P_i'', and we have

$$P = \Sigma(P_i'' - P_i').$$

Hence,

$$\Gamma \sim (H) + \Sigma(P_i'' - P_i')$$

where, moreover, the two cycles are reducible to each other by a finite deformation.

Now, let $\bar{\gamma}_i$ be the projection of γ_i into H_{y_1} assumed in K_{a_i} and such that $|y_1 - a_0| > |a_i - a_0|$, then slide γ_i along its projecting lines until it coincides with $\bar{\gamma}_i$. During the motion P_i' and P_i'' are deformed assuming final positions Q_i', Q_i'', hence

$$(P_i'' - P_i') - (Q_i'' - Q_i') \sim 0.$$

Now $Q_i'' - Q_i'$ may be generated by a certain cycle δ of H_y when y starting from a_0 describes a closed path surrounding once a_i, the nature of this path being on the whole immaterial. Denoting by δ_i the same linear cycle of H_y as in § 4, and by Δ_i its two-dimensional locus when y describes the straight path a_0a_i, let us then make y_1 tend towards a_i on the line

$$\text{argt. } (y - a_0) = \text{argt. } (a_i - a_0).$$

When y describes the closed path around a_i, δ undergoes an increment of $\lambda_i\delta_i$, and when y_1 coincides with a_i, $Q_i'' - Q_i'$ will have become $\lambda_i\Delta_i$ hence

$$Q_i'' - Q_i' - \lambda_i\Delta_i \sim 0; \qquad \therefore \Gamma \sim \Sigma\lambda_i\Delta_i + (H_0).$$

The cycle at the right is at finite distance, hence at once

$$-\Sigma\lambda_i\delta_i \equiv (H_0), \qquad (\text{mod } H_0).$$

Conversely if this last congruence is verified, that is if $\Sigma\lambda_i\delta_i$ bounds a finite portion of H_0, $\Sigma\lambda_i\Delta_i + (H_0)$ is a two-dimensional cycle without infinite points. This is an immediate consequence of the congruences $\Delta_i \equiv \delta_i$. Thus: *A finite cycle is reducible by finite deformation to a sum of multiples of the (Δ)'s plus a part of H_0. The linear cycle $\Sigma\lambda_i\delta_i$ bounds a finite part of H_y and conversely to such a cycle corresponds a finite cycle $\Sigma\lambda_i\Delta_i + (H_0)$.*

This very important theorem was discovered by Picard* the proof here given being a modification of one due to Poincaré.†

14. A plane section or in fact any irreducible algebraic curve C forms a two-dimensional cycle. Such a cycle cannot be dependent upon finite cycles. For if Γ is finite $(\Gamma, H) = (\Gamma, H_\infty) = 0$, while on the contrary $(C, H) \neq 0$ (No. 12).

Since H is independent of the finite cycles and on the other hand whatever Γ there is a finite cycle $\sim m\Gamma + nH$, *the number of independent finite cycles is precisely equal to $R_2 - 1$.*

§ 7. *Formula for the index R_2.*

15. The number of independent finite cycles may also be derived in a different manner and this will lead us to the formula for R_2. We have seen that when a finite cycle Γ is reduced to the type $\Sigma\lambda_i\Delta_i + (H_0)$ then $\Sigma\lambda_i\delta_i$ assumed in H_0 bounds a finite part of it, which we express by

$$\Sigma\lambda_i\delta_i \equiv -(H_0) \tag{2}$$

and we must first determine the number of independent congruences (2).

Let γ be any linear cycle of H_0 bounding an M_2 of F. By a discussion similar to that of Nos. 13, 14 we may show that there is an $M_2' = nM_2 + \Gamma_1'$, ($n \neq 0$, Γ' two-dimensional cycle) meeting nowhere the curves H_i, H_∞ and bounded by $n\gamma$. From this will follow a congruence

$$M_2' - \Sigma\lambda_i\Delta_i + (H_0) \equiv 0; \qquad \therefore n\gamma - \Sigma\lambda_i\delta_i \equiv -(H_0),$$

that is if $\gamma \sim 0$ mod. F, there is a finite part of H_0, *bounded by a combination of the cycles γ, δ_i.* The congruences $\delta_i \equiv \Delta_i$ show that the (δ)'s themselves form boundary on F. Hence the number of (δ)'s which do not satisfy a congruence (2) is exactly equal to the number of zero cycles

* Picard-Simart, vol. II, p. 335.

† Second Mémoire, p. 155, §§ 4, 5.

of F contained in H_0 of which no combination forms the boundary of a finite part of H_0, that is finally to $2p - 2q + m - 1$.* As there are N cycles δ_i, the number of distinct congruences (2) is $N-(2p-2q+m-1)$.

16. To each congruence (2) corresponds a two-dimensional cycle

$$\Gamma = \Sigma\lambda_i\Delta_i + (H_0).$$

We have to find how many of these cycles are ~ 0 though corresponding to distinct congruences.

Assume that Γ corresponding to (2) bounds an M_3 and let γ be the linear cycle intersection of M_3 and H_y. M_3 can be considered as the locus of γ when y describes the y plane without crossing the cuts a_0, a_i. The boundaries of M_3 are obtained when y describes the lips of these cuts.

Now the congruence (2) shows that when y describes the lips of the cut a_0a_i the difference of the two manifolds generated by γ is $\lambda_i\Delta_i$ and therefore when y turns around a_i, γ receives the increment $\lambda_i\delta_i$. Hence if the (λ)'s are not all zero, γ *is not an invariant cycle.*

Let Γ' be a tube contained in M_3 and having for axis the position of γ in H_∞. If we suppress in M_3 the portion within Γ' we obtain a finite M_3' bounded by Γ and Γ'. Thus if $\Gamma \sim 0$ *it bounds together with a tube of axis in H_∞ a certain finite M_3.*

Conversely let γ be a linear cycle of H_y and consider its locus Γ' when y describes a circle of large radius. $\Gamma' \sim 0$ and moreover it will meet nowhere the curves H_i, H_0, if as we may arrange it γ does not go through a base point of $\{H_y\}$. Let us deform the circle until it coincides with the lips of the cuts a_0a_i. There will be a corresponding finite deformation of Γ' reducing it ultimately to a cycle $\Gamma = \Sigma\lambda_i\Delta_i + (H_0)$. $\lambda_i\Delta_i$ indicates the difference between the manifold generated by γ on both sides of a_0a_i and therefore $\lambda_i\delta_i$ is the increment of γ when y turns around a_i. Hence if γ is not an invariant cycle the (λ)'s are not all zero and we have here a finite two-dimensional zero cycle relative to a congruence (2) with coefficients λ_i not all zero. The number of distinct congruences corresponding to such cycles is therefore equal to the number of distinct cycles of H_y of which no combination is invariant, that is to $2p - 2q$.

17. It follows from the preceding discussion that there are

$$N - (4p - 4q + m - 1)$$

distinct finite cycles. Equating this to the number $R_2 - 1$ already found we obtain

$$R_2 = N - (4p + m) + 4q + 2.$$

* Poincaré assumed this number to be $2p + m - 1$ hence did not arrive at the correct value for R_2. The correct value here given was obtained by Picard by transcendental methods. See Picard-Simart, vol. II, pp. 377, 398.

The number $I = N - (4p + m)$ formed with the characteristic numbers N, m, p of $|H|$ is a numerical invariant which would remain the same if it were computed from the corresponding characters of a linear system such as $|E|$ of No. 6. It is in fact known as the invariant of Zeuthen-Segre. Various geometers (Segre, Castelnuovo and Enriques) have shown how it may be computed from the characters of any linear pencil of curves. Be it as it may we have the fundamental formula

$$R_2 = I + 4q + 2,$$

in which enter only the invariants of F. It was first correctly derived by Alexander (loc. cit.) by two distinct and very original methods based essentially on a subdivision of F into a generalized polyhedron, much as is done by Poincaré in his first Mémoire, followed by an application of the Euler-Poincaré polyhedron formula.

Integrals belonging to algebraic surfaces.

§ 1. *Double integrals and their residues.*

19. Given a rational point function $R(x, y, z)$ on F we may set up a double integral

(1) $$\int\int R(x, y, z)dxdy.$$

This integral is assumed taken over an M_2 of the surface. Just as with abelian integrals we are here led to integrate over cycles at all points of which the integral is finite. The corresponding value of (1) is called a *period*. A cycle may be reducible by deformation to a line and yet yield a period. It is then a tubular manifold having for axis this line which must be situated in the curve of discontinuity of (1). The period is then said to be a *residue*.

19. The following theorem will prove useful in the study of residues.* Here and in the sequel we denote throughout by $\partial U/\partial x$, $\partial U/\partial y$, the partial derivatives of $U(x, y, z)$ when z is considered as a function of x, y, reserving the notation U_x', U_y', U_z', for the partial derivatives when the variables are all to be considered as independent. Then, *if at all points of a two-dimensional cycle* Γ, *the rational function* $U(x, y, z)$ *is finite and* $F_z' \neq 0$

$$\int\int_\Gamma \frac{\partial U}{\partial x} dxdy = \int\int_\Gamma \frac{\partial U}{\partial y} dxdy = 0.$$

Let us take for example the first integral. At all points of Γ

$$\frac{D(x, y)}{D(U, y)} = \frac{F_z'}{U_x'F_z' - U_z'F_x'} \neq 0.$$

* It was first given in the writer's Mémoire in the Annali di Mat., ser. III, vol. 36 (1917), § 8.

Hence the change of variables $X = U(x, y, z)$, $Y = y$ transforms Γ into a finite cycle Γ' of the real four-space image of the values of X, Y. Applying Poincaré's extension of Cauchy's fundamental theorem to functions of two complex variables we obtain at once

$$\int\int_{\Gamma} \frac{\partial U}{\partial x} dxdy = \int\int_{\Gamma'} dXdY = 0,$$

as was to be proved.

20. A first corollary is that the integrals

$$(2) \qquad \int\int \frac{\partial U}{\partial x} dxdy, \qquad \int\int \frac{\partial U}{\partial y} dxdy,$$

have no residues. Indeed a cycle corresponding to a residue may always be deformed so as to satisfy the conditions of the theorem.

Let now D be a curve of discontinuity of $R(x, y, z)$ other than the curve at infinity and $g(x, y) = 0$ its projection in the xy plane. We may write

$$R(x, y, z) = \frac{A(x, y, z)}{[g(x, y)]^{\alpha}},$$

where $A(x, y, z)$ is a rational function finite on D and α is a positive integer. Assuming as we may if the axes are properly chosen that $g(x, y)$ contains the variable y, we have identically if $\alpha > 1$,

$$\int\int \left(\frac{A}{g^{\alpha}} + \frac{1}{\alpha - 1} \frac{\partial}{\partial y} \frac{A}{g_y' g^{\alpha-1}}\right) dxdy = \int\int \frac{1}{(\alpha - 1) g^{\alpha-1}} \frac{\partial}{\partial y} \left(\frac{A}{g_y'}\right) dxdy.$$

This coupled with the remark made above shows that as far as its residues with respect to D are concerned the integral

$$(1) \qquad \int\int R(x, y, z) dxdy = \int\int \frac{A(x, y, z)}{[g(x, y)]^{\alpha}} dxdy$$

can be replaced by one of the same type with $\alpha - 1$ in place of α. This can be continued until we arrive at an integral of type

$$(3) \qquad \int\int \frac{A(x, y, z)}{g(x, y)} dxdy$$

and with the same residues as the initial one.

Let now Γ be a tube of axis γ in D corresponding to a residue of (1) and so deformed as to have no points at infinity. Passing from the variables of integration x, y, to the variables x, g, and denoting by Γ' the transformed of Γ in the real four-space corresponding to the last pair of variables, the residue of (3) relatively to Γ is equal to the value of

$$(4) \qquad \int\int_{\Gamma'} \frac{A(x, y, z)dxdg}{gg_y'}.$$

Now Γ' is a closed tube of axis γ' in $y = 0$. Let P be a point of γ', $x_1' + ix_1''$ its x. The real three-space $x' = x_1'$ of the complex (x, g) space intersects the manifold $g = 0$ in a line δ through P and Γ' in a closed circuit ϵ surrounding δ. We can slide ϵ in its space until it becomes a circuit ϵ' of the plane $x' = x_1'$ and this without its ever meeting δ. Carrying the process out as P describes γ will result in deforming Γ' into a tube Γ'' generated by a circuit ϵ' of the planes $x = \text{Ct.}$, when x describes a certain circuit ζ in its plane. As the deformation from Γ' to Γ'' may be made without ever meeting $g = 0$ we have

$$\int\int_{\Gamma'} \frac{Adxdg}{gg_y'} = \int_{\zeta} \frac{Adx}{g_y'} \int_{\epsilon'} \frac{dg}{g} = 2\pi i \int_{\gamma} \frac{Adx}{g_y'}$$

which shows that the *residues of* (3) *are the periods of the abelian integral*

$$(5) \qquad 2\pi i \int \frac{A(x, y, z)dx}{g_y'}, \qquad g = 0$$

belonging to the curve D.

The converse of this proposition is almost evident for we can easily build a continuum Γ' corresponding to a period of (5) and obtain from it a tube Γ.

21. Residues may be classified into two kinds according to the type of residues which they constitute for the corresponding integrals. A *point residue* corresponds to a logarithmic period and a *cyclic residue* to a cyclic period. To a point residue relatively to D corresponds a tube of axis drawn around certain points of the curve—points which are either intersections with other discontinuities or else *accidental* singularities, that is singularities such as the point of contact with a tangent plane for the section by it. Hence if the integral has only one curve of discontinuity deprived of accidental singularities it has no point residues. Finally from a well-known property of logarithmic periods of abelian integrals follows that the point residues corresponding to any curve of discontinuity have a zero sum. We may mention that the residues corresponding to a given point of the surface also have a zero sum.*

§ 2. *Double integrals of the second kind.*

22. The classification of abelian integrals on the basis of their discontinuities may be extended to double integrals. We say with Nöther that

* See the writer's note in the Quarterly Journal, vol. 47 (1917), pp. 333–348.

$$\int\int R(x, y, z)dxdy \tag{1}$$

is of the *first kind* if it is finite everywhere. It is of the *second kind* if it behaves in the vicinity of any curve of discontinuity like one of type

$$\int\int\left(\frac{\partial U}{\partial x}+\frac{\partial V}{\partial y}\right)dxdy, \tag{6}$$

where U, V are rational in x, y, z.* The difference between (1) and (6) being finite in the vicinity of the curve it follows from No. 20 that (1) has no residues and as we shall see this property is characteristic.

Two integrals of the second kind differing only by one of type (6) are said to be *equivalent*. Our fundamental problem consists in *finding the number of inequivalent integrals of the second kind.*

An integral of the second kind is said to be *improper* if of type (6), *proper* if not of this type.

An integral is of the *third kind* if it belongs to neither of the two kinds already mentioned. Our discussion will be wholly confined to integrals of the second kind.

23. We proceed to show that integrals without residues are of the second kind. For if (1) is infinite on a curve D of projection $g(x, y) = 0$ we may by subtracting a suitable integral (6) reduce it to the form

$$\int\int\frac{A(x, y, z)}{g}dxdy.$$

The residues which are the periods of

$$2\pi i\int\frac{A(x, y, z)dx}{g_y'}, \qquad g(x, y) = 0,$$

being all zero, this abelian integral is a rational function $2\pi i\cdot U(x, y)$ on D, hence

$$\begin{aligned}\int\int\left[\frac{A(x, y, z)}{g}-\frac{(U_x'g_y'-U_y'g_x')}{g}\right]dxdy\\ =\int\int\left[\frac{A}{g}-\frac{D\left(\frac{U}{g}, g\right)}{D(x, y)}\right]dxdy\\ =\int\int\frac{A}{g}dxdy-\int\int\frac{\partial}{\partial x}\left(\frac{U}{g}\right)dxdg\end{aligned}$$

is finite in the vicinity of D, which is sufficient to show that (1) behaves

* This definition differs slightly from Picard's, but seems somewhat more convenient. See Picard-Simart, vol. II, p. 160.

like an integral (6) in the neighborhood of D. Obviously then when there are no residues the integral is of the second kind.*

24. Let us now prove *that integrals of type* (6) *preserve their form under birational transformations.* Indeed we have

$$\frac{\partial U}{\partial x}+\frac{\partial V}{\partial y}=\frac{D(U,y)}{D(x,y)}+\frac{D(x,V)}{D(x,y)};$$

$$\frac{D(U,V)}{D(x,y)}=\frac{\partial}{\partial x}\left(U\frac{\partial V}{\partial y}\right)-\frac{\partial}{\partial y}\left(U\frac{\partial V}{\partial x}\right).$$

Hence if x', y', z' are the new variables and U_1, V_1 rational functions of them

$$\int\int\left(\frac{\partial U}{\partial x}+\frac{\partial V}{\partial y}\right)dxdy=\int\int\left(\frac{D(U,y)}{D(x',y')}+\frac{D(x,V)}{D(x',y')}\right)dx'dy'$$
$$=\int\int\left(\frac{\partial U_1}{\partial x'}+\frac{\partial V_1}{\partial y'}\right)dx'dy'$$

as we wished to show. It follows in particular that if an integral (1) behaves like an integral of the second kind in the vicinity of a curve D of F not fundamental for the transformation, its transformed will behave similarly in the vicinity of the transformed curve of D on the transformed surface.

§ 3. *On the periods of certain double integrals.*

25. The reduction and enumeration of integrals of the second kind rests upon the study of the periods of a class of double integrals of which the simplest are of type

$$\int\int\frac{Q(x,y,z)}{F'_z}dxdy, \tag{7}$$

where Q is an adjoint polynomial of the surface. Such an integral has a meaning whenever taken over a manifold at finite distance for the integrand is then never infinite to a power greater than one half. Closely associated to (7) is the abelian integral of H_y

$$\int\frac{Q(x,y,z)}{F'_z}dx. \tag{8}$$

Let $\Omega_i(y)$ be the period of (8) with respect to the cycle δ_i of No. 6, unchanged in the vicinity of the critical point a_i. As (7) extended to a finite portion of H_0 is zero, its value extended to a cycle $\sum\lambda_i\Delta_i+(H_0)$ such as considered in Part I, is equal to

* Picard-Simart, vol. II, p. 203.

$$\sum \lambda_i \int_{a_0}^{a_i} \Omega_i(y)dy, \tag{9}$$

and the condition $\sum \lambda_i \delta_i \equiv (H_0)$, gives here since a_0 is arbitrary

$$\sum \lambda_i \Omega_i(y) = 0. \tag{10}$$

26. If we impose upon a polynomial $g(x, z)$ the condition of being adjoint to the curve $F(x, y, z) = 0$ we merely constrain its coefficients to satisfy a certain number of linear equations with coefficients rational in those of F, at least this will certainly be the case for reasons of symmetry if the curve has no other singularities than double points with distinct tangents. Hence the most general polynomial of a given order adjoint to the curve is a linear combination of a certain number of polynomials with coefficients rational in those of the curve, or in the last analysis of a certain number of polynomials adjoint to the surface and of the given degree in x and z alone if not in the three variables x, y, z. As a consequence (No. 5), an arbitrary integral (8) has $2p + m - 1$ distinct periods and we may give ourselves $2p + m - 1$ such integrals with a non-zero period-determinant of order equal to their number.

27. Starting now with the same adjoint polynomial $Q(x, y, z)$ that appears in (7) and with a polynomial in y, $\varphi(y)$, we form the double integral

$$\int\int \frac{\varphi(y)Q(x, y, z)}{F_z'} dxdy, \tag{11}$$

and we shall show that its periods may take completely arbitrary values. These periods are quantities.

$$\sum_j \lambda_j^h \int_{a_0}^{a_j} \varphi(y)\Omega_j(y)dy, \qquad \sum_j \lambda_j^h \Omega_j(y) = 0.$$

If our affirmation were incorrect there would have to exist a relation such as

$$\sum_j \mu_j \int_{a_0}^{a_j} y^k \Omega_j(y)dy = 0$$

with coefficients μ_j not all zero, satisfied for all integral values of the exponent, k. This would lead at once to

$$\sum_j \mu_j \int_{a_0}^{a_j} \frac{\Omega_j(y)}{y - a} dy = 0$$

for a arbitrary. Assuming then $\mu_j \neq 0$ the increment $2\pi i \mu_j \Omega_j(y)$ of the left-hand side when y describes a closed circuit surrounding a_j would have to be zero. But in that case $\Omega_j(y) = 0$ and as a consequence the

cycles of H_y would all be invariant in the vicinity of a_j, (No. 6), which is impossible if F is not ruled or of Steiner and $\{H_y\}$ is arbitrary. We conclude then that the periods of an arbitrary integral (7) may have any set of $N - (2p + m - 1 - R_1)$ values. As we have seen $2p - R_1$ of the corresponding cycles are reducible by finite deformation to tubes with axes in H_∞, manifolds which correspond to residues of the double integral. We can form integrals (7) for which these residues are equal to zero, the other periods having arbitrary values, and these integrals are of the second kind.

28. Nearly all this applies to integrals of the second kind,

$$(12) \qquad \int\int \frac{Q(x, y, z)}{\varphi(y)F_z'} dxdy,$$

where Q is adjoint to F and the polynomial φ has no critical value for root. The period corresponding to (9) is still

$$\sum \lambda_j \int_{a_0}^{a_j} \Omega_j(y)dy,$$

where now Ω_j is the period of

$$(13) \qquad \int \frac{Q(x, y, z)}{\varphi(y)F_z'} dx$$

relatively to δ_i. The value of this period does not change when the cuts are made to cross a root y_0 of $\varphi(y) = 0$. Indeed when for example a_0a_j is made to cross y_0 the period increases by $2\pi i$ times the residue of $\Omega_j(y)$ with respect to the pole y_0. But this increment must be zero for its value is also the residue of the double integral of the second kind (13) with respect to a tube having for axis the position of δ_j in H_{y_0}. Similarly if $\omega(y)$ is any period of (13), the function

$$\int \omega(y)dy$$

is meromorphic in the neighborhood of the roots of $\varphi(y)$, and in particular if $\omega(y)$ is rational the integral is also a rational function of y. All this ceases to be true when either (12) is not of the second kind or $\varphi(y)$ vanishes at a critical point.

Remark. We have just mentioned periods of (13) rational in y. An example is furnished by a period corresponding to an invariant cycle, for it is meromorphic for all values of y.

29. *If the integral* (12) *is without periods with respect to the finite cycles it can be put in the form*†

* See Picard-Simart, vol. II, p. 354.

† The proof here given is not essentially different from that in Picard-Simart, vol. II, p. 365, for the special case $\varphi = 1$.

$$\iint \left(\frac{\partial}{\partial x} \frac{A}{\varphi F_z'} + \frac{\partial}{\partial y} \frac{B}{\varphi F_z'} \right) dx dy \tag{14}$$

(A, B, adjoint polynomials, φ polynomial in y). Let us recall first that as shown in No. 26 we may choose $2p + m - 1$ integrals

$$\int \frac{Q_j(x, y, z)}{F_z'} dx \quad (j = 1, 2, \cdots, 2p + m - 1), \tag{15}$$

with a non-zero determinant of periods. Let $\gamma_1, \gamma_2, \cdots, \gamma_{2q}$ be $2q$ invariant cycles of H_y independent mod F, and denote by ω_j, ω_{hj} the periods of (13) and (15) with respect to γ_j, and by Ω_{hk} that of (15) with respect to δ_k. We may consider the equations in the (c)'s

$$\int_{a_k}^{y} \Omega_k(y) dy = \sum_{h=1}^{2p+m-1} c_h \Omega_{hk}(y) \qquad (k = 1, 2, \cdots, N);$$

$$\int_{a_0}^{y} \omega_j(y) dy = \sum_{h=1}^{2p+m-1} c_h \omega_{hj}(y) \qquad (j = 1, 2, \cdots, 2q).$$

Since any cycle of H_y together with the (γ)'s and the (δ)'s can always be so combined as to form the boundary of a finite part of H_y the matrix of the coefficients is of rank $2p + m - 1$. The distinct combinations of the right-hand sides identically zero are all derived from the relations

$$\sum_{k=1}^{N} \lambda_k \Omega_{hk}(y) = 0 \quad (\lambda_k \text{ integer}; \ h = 1, 2, \cdots, 2p + m - 1),$$

which are known to exist and there are $N - (2p - 2q + m - 1)$ of them. But when such a relation is satisfied, $\sum \lambda_k \delta_k$ bounds a finite part of H_y and $\sum \lambda_k \Delta_k + (H_0)$ is a two-dimensional cycle without infinite points. The corresponding period of (12) is then zero by assumption, hence

$$\sum \lambda_k \int_{a_0}^{a_k} \Omega_k(y) dy = 0,$$

or since a_0 is arbitrary and can be replaced by y,

$$\sum \lambda_k \int_{a_k}^{y} \Omega_k(y) dy = 0.$$

This shows that the equations in the (c)'s are compatible, and therefore furnish a unique solution. As the left hand sides and the coefficients are meromorphic for arbitrary values of y as well as for the roots of $\varphi(y)$ (No. 28) the same is true for the (c)'s. The only singularities which they might acquire are the critical values. But when y describes a closed circuit around a_j the quantities

$$\int_{a_k}^{y} \Omega_k(y)dy, \qquad \Omega_{hk}(y),$$

receive increments

$$\lambda \int_{a_j}^{y} \Omega_j(y)dy, \qquad \lambda\Omega_{hj}(y),$$

and therefore the system of equations which we are considering remains invariant. This shows that even a critical value is at most a pole of the (c)'s so that these functions are rational in y.

Setting

$$\sum_{h=1}^{2p+m-1} c_h Q_h(x, y, z) = \frac{A(x, y, z)}{\psi(y)},$$

where A and ψ are polynomials, the first adjoint to F, and considering

$$\int \left(\frac{Q(x, y, z)}{\varphi(y)F_z'} - \frac{\partial}{\partial y} \frac{A(x, y, z)}{\psi(y)F_z'} \right) dx$$

we remark that it reduces to a rational function on H_y, hence

$$\frac{Q(x, y, z)}{\varphi(y)F_z'} - \frac{\partial}{\partial y} \frac{A(x, y, z)}{\psi(y)F_z'} = \frac{\partial U}{\partial x},$$

where U is a rational function. As it can have no poles at a finite distance for y fixed,

$$U = \frac{B(x, z)}{\psi(y)F_z'}.$$

Since A, B can be replaced by $A\psi_1$, $B\psi$, and ψ, ψ_1 by their product, our proof is now complete.

§ 4. *Integrals of the second kind. Reduction and enumeration.*

30. Let D be an irreducible algebraic curve in general position on F, $g(x, y) = 0$ its projection,

$$R(x, y, z) = \frac{A(x, y, z)}{B(x, y, z)} \qquad (A, B \text{ polynomials}),$$

a rational function infinite on the curve. The resultant of B and F considered as polynomials in z alone is of the form $g^\alpha \cdot G(x, y)$ with G a polynomial prime to g, and we have

$$g^\alpha \cdot G = C_1 \cdot D_1 + D_2 \cdot F, \qquad (C_1, D_1, D_2, \text{ polynomials}).$$

Hence on F

$$R = \frac{AC_1}{g^\alpha \cdot G} = \frac{A_1}{g^\alpha \cdot G}.$$

On the other hand, breaking into partial fractions,

$$\frac{1}{g^\alpha \cdot G} = \frac{H_1(x, y)}{\varphi(y) \cdot g^\alpha} + \frac{H_2(x, y)}{\varphi(y) \cdot G(x, y)}, \quad (H_1, H_2, \text{ polynomials}).$$

Hence in the vicinity of $g(x, y) = 0$, R is infinite like

$$R_1(x, y) = \frac{A_1 H_1}{\varphi(y) \cdot g^\alpha} = \frac{K(x, y, z)}{\varphi(y) \cdot g^\alpha},$$

and as $\varphi(y)$ vanishes only for the (y)'s of the points for which $g = G = 0$, and the curve D is in general position, φ has no critical value for root. We may note in particular that if $g(x, y) = 0$ intersects F into two curves on one of which R is finite, R_1 will be finite on the same curve.

31. Given an arbitrary integral of the second kind with curves of discontinuity in general position,

$$\int\int R(x, y, z)dxdy = \int\int F_z' \cdot R(x, y, z)\frac{dxdy}{F_z'} \tag{16}$$

we may apply the first part of the preceding discussion to RF_z' and put the integral in the form

$$\int\int \frac{A(x, y, z)dxdy}{\prod_i [g_i(x, y)]^{\alpha_i} F_z'}, \tag{17}$$

where $g(x, y) = 0$ is the projection of an irreducible curve of discontinuity D_i, the integral being finite on the other curve of F having the same projection. Having effected if necessary a preliminary transformation of coördinates we are at liberty to assume that the (D)'s occupy a general position with respect to the axes.

The double integral behaves in the vicinity of D_i like one of type

$$\int\int \left(\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y}\right) dxdy.$$

In so far as the vicinity of D_i is concerned this last integral, according to what we have just seen, may be replaced by another

$$\int\int \left(\frac{\partial}{\partial x}\frac{C_1}{\varphi(y)g_i^{\beta_i}} + \frac{\partial}{\partial y}\frac{C_2}{\varphi(y)g_i^{\beta_i}}\right) dxdy, \tag{18}$$

where C_1, C_2, φ are polynomials, $\varphi(y)$ having no critical value for root. Subtracting from (17) all the integrals (18) corresponding to its various curves of discontinuity we obtain one of form

$$\int\int \frac{A_1(x, y, z)dxdy}{\varphi(y)\prod_i g_i^{\alpha_i}(x, y)F_z'}$$

equivalent to (17) but infinite only at infinity and on the curves $\varphi(y) = 0$, of which none correspond to critical values. Clearly the polynomial $A_1(x, y, z)$ must vanish at least once on the complete intersection of F with $g_i = 0$ if it is a simple curve of F, and at least twice if it is double. The (g)'s being assumed irreducible, we can apply Nöther's generalized theorem* which gives

$$A_1 = Ag_i + BF \qquad (A, B \text{ polynomials}).$$

Hence on the surface

$$\frac{A_1}{g_1} = A, \text{ polynomial},$$

so that the integral may be replaced by one of the same form with $\alpha_i - 1$ in place of α_i. This can be continued indefinitely and for all the (g)'s so that the integral assumes finally the form

$$\text{(19)} \qquad \int\int \frac{Q(x, y, z)dxdy}{\varphi(y)F_z'} \qquad (\varphi \text{ has no critical value for root})$$

and as it is finite on the double curve the polynomial $Q(x, y, z)$ is adjoint to the surface.

32. According to No. 27 we may form an integral

$$\text{(20)} \qquad \int\int \frac{P(x, y, z)dxdy}{F_z'} \qquad (P \text{ adjoint polynomial})$$

with periods equal to those of (19). Their difference is of the same type as (19) and therefore (No. 29) improper of the second kind, hence (19) and (20) are equivalent. We conclude then from this that *any integral of the second kind is equivalent to one of form* (20) *that is to an integral finite at finite distance*.† Since H_∞ is an arbitrary H we may affirm *that any integral of the second kind is equivalent to one infinite on an arbitrary plane section and nowhere else*. This shows that such an integral behaves as if it were improper in the neighborhood of *any* point of F, and not merely so in the neighborhood of a generic point of the curves of discontinuity as the definition simply asserts. As a corollary when F is birationally transformed into a surface with ordinary singularities F', an integral of the second kind J of F becomes one of the second kind J' of F'. For if J_1 is the improper integral corresponding to a point M transformed into a point M' or into a curve M'' (fundamental curve) of F' and J_1' the integral into which J_1 is changed, the difference $J - J_1 = J' - J_1'$ being finite in the vicinity of M on F, will also remain finite

* Picard-Simart, vol. II, p. 17.

† Picard-Simart, vol. II, p. 186.

in that of M' or of a generic point of M'' on F and as J_1' is improper of the second kind for F' our assertion is justified.

Remark. Integrals of the second kind belonging to a surface F with arbitrary singularities may be defined as those which become of the second kind when F is transformed into a surface with ordinary singularities.

33. With R_2 integrals of the second kind of type (20) we can always form a linear combination without periods and therefore improper. Hence the number ρ_0 of unequivalent integrals of the second kind is given by

$$\rho_0 = R_2 - \rho = I + 4q_1 - \rho + 2; \qquad (\rho \text{ integer } \geqq 1),$$

which is Picard's classical formula.* In the next section the number ρ will be identified with the one that he has designated by the same letter. According to No. 28 ρ_0 remains invariant under birational transformations. As is well known this is also the case for q but not for I and therefore not for R_2 or ρ. In the customary terminology ρ_0, q are absolute invariants and ρ, I, R_2, are relative invariants.

§ 5. *Integrals of total differentials.*

34. An integral of total differentials is an integral such as

$$\int U dy + V dx, \tag{21}$$

where U, V, are rational functions of F satisfying the condition of integrability of Poincaré

$$\frac{\partial U}{\partial x} = \frac{\partial V}{\partial y}$$

thanks to which (21) is a point-function on the surface. The integral has *periods* obtained by taking for paths of integration linear cycles of F. If the cycle reduces to a small circuit around a curve of discontinuity the period is called *logarithmic,* otherwise it is called *cyclic.* When the small circuit is displaced without intersecting the curves of discontinuity the logarithmic period is unchanged so that it really belongs to the curve of discontinuity considered, also called *logarithmic curve.* As any linear cycle is homologous to one in a given plane section, the periods of (21) are also those of

* See Picard-Simart, vol. II, p. 497. This formula holds even for a ruled or a Steiner surface. Indeed the whole argument holds with only slight modifications if we replace everywhere $|H|$ by $|2H|$, this being also true for the next section. As a matter of fact for a ruled surface

$$N = m, \qquad R = 2p, \qquad \rho = 2 \text{ (Severi)}, \qquad \therefore \rho_0 = 0.$$

That is, a ruled surface has no proper integrals of the second kind. This holds also for a Steiner surface since it is rational.

$$(22) \qquad \int U dy, \qquad \int V dx$$

belonging to H_x, H_y respectively.

Integrals of total differentials have been divided into three kinds on the same basis as abelian integrals. The first kind includes integrals finite everywhere; the second, integrals without logarithmic curves; and the third, integrals possessing some.

Let $C_1, C_2, \cdots, C_\nu$ be irreducible logarithmic curves of (21), m_i, μ_i the order of C_i and the corresponding logarithmic period. The integrals (22) will have m_i logarithmic points with a period of μ_i for each, therefore $\sum_{i=1}^{\nu} m_i\mu_i = 0$. This shows that an integral of the third kind has at least *two* logarithmic curves. In point of fact we shall show in the next section that $\rho + 1$ is precisely the least number of logarithmic curves of such an integral.

The study of integrals of the first kind forms one of the most interesting chapters of algebraic geometry, but as it is unrelated to the considerations of this paper we shall omit it here.

§ 6. *Integrals of total differentials of the second kind.*

35. In the notations of No. 25 we may consider the system of equations in the unknowns c,

$$\sum_{h=1}^{2p-m-1} c_h\Omega_{hj}(y) = 0; \qquad \sum_{h=1}^{2p-m-1} c_h\omega_{hk}(y) = d_k;$$
$$(j = 1, 2, \cdots, N;\ k = 1, 2, \cdots, 2q),$$

where the (d)'s are arbitrary constants not all zero. It may be shown here again that there is a unique solution composed of rational functions of y. The integral

$$\int V dx \qquad \left(V = \sum \frac{c_h Q_h(x, y, z)}{F_z'}\right),$$

has only the (d)'s for periods, hence

$$(23) \qquad \frac{\partial}{\partial y}\int V dx$$

is a rational function on H_y. Reasoning as before, we may show that there exists a rational function $U(x, y, z)$ such that

$$\frac{\partial U}{\partial x} = \frac{\partial V}{\partial y},$$

so that

$$\int U dy + V dx$$

is an integral of total differentials. It may have logarithmic curves belonging to $\{H_y\}$ but these may be eliminated, as we have had occasion to show, by subtracting integrals

$$\int \frac{\alpha dy}{y - a},$$

which do not affect the periods with respect to the linear cycles of F, periods equal to those of (25) that is to the (d)'s. The integral finally obtained is said to be *of the second kind.* Obviously we may have $2q$ such integrals with no linear combination deprived of periods, that is with no linear combination equal to a rational function, which is expressed by the statement that *there are $2q$ linearly independent integrals of total differentials of the second kind.* Let us mention that *the number of linearly independent integrals of total differentials of the first kind is equal to q.*

§ 7. *Integrals of total differentials of the third kind and their relation to ρ.*

36. We have seen that there are $R_2 - 1$ distinct integrals of type (19) with non-zero periods. Hence if $\rho = R_2 - \rho_0 > 1$, $\rho - 1$ of these integrals must be improper, that is there must be $\rho - 1$ distinct improper integrals of type (19) with non-zero periods. When (19) is improper

$$\frac{Q(x, y, z)}{\varphi(y)F_z'} = \frac{\partial U}{\partial x} + \frac{\partial V}{\partial y}$$

and according to Nos. 30, 31 and with similar notations U, V are both of the form

$$\frac{A(x, y, z)}{\prod_i [g_i(x, y)]^{\alpha_i} \cdot F_z'}.$$

The logarithmic periods of

(24) $$-\int V dx$$

relatively to the points where H_y meets D_i, irreducible curve of infinity of U, V, not contained in $\{H_y\}$, are functions of y alone. But their derivatives with respect to y are zero, hence they have a constant value μ_i, the same for all points on D_i. Now if (19) has no periods

$$\frac{Q(x, y, z)}{\varphi(y)F_z'} = \frac{\partial U_1}{\partial x} + \frac{\partial V_1}{\partial y}$$

where U_1, V_1 are both of the form

(24) $$\frac{A(x, y, z)}{\psi(y) \cdot F_z'}.$$

* See Picard-Simart, vol. I, p. 100.

Comparing we obtain

$$\frac{\partial(U - U_1)}{\partial x} + \frac{\partial(V - V_1)}{\partial y} = 0,$$

hence

(25) $$\int (U - U_1)dy - (V - V_1)dx$$

is of total differentials. Let H_{y_k} (y_k finite), be a curve of discontinuity of (25), $2\pi i \alpha_k$ the corresponding logarithmic period. *The integral*

$$\int \left(U - U_1 - \sum \frac{\alpha_k}{y - y_k} \right) dy - (V - V_1)dx$$

has no other logarithmic curves than H_∞ and the (D's), the period relatively to D_i being μ_i.

37. Conversely let us assume the existence of an integral

$$\int U_2 dy - V_2 dx, \qquad \frac{\partial U_2}{\partial x} + \frac{\partial V_2}{\partial y} = 0$$

such as we have just described. As we may write

$$\frac{Q(x, y, z)}{\varphi(y)F_z'} = \frac{\partial(U - U_2)}{\partial x} + \frac{\partial(V - V_2)}{\partial y}$$

the integral

$$- \int (V - V_2)dx$$

behaves like a rational function everywhere on H_y except perhaps at infinity. As its points of discontinuity are determined rationally in terms of y, there exists a rational function $R(x, y, z)$ such that

$$V_1 = V - V_2 - \frac{\partial R}{\partial x}$$

is of the form

$$\frac{A(x, y, z)}{\varphi(y)F_z'}.$$

Then owing to the identity

$$\frac{Q(x, y, z)}{\varphi(y)F_z'} = \frac{\partial U_1}{\partial x} + \frac{\partial V_1}{\partial y}, \qquad U_1 = U - U_2 + \frac{\partial R}{\partial y},$$

U_1 is of the same form as V_1 and finally (19) has no periods. Hence, *in order that the double integral (19) be without periods it is necessary and sufficient that there exist an integral of total differentials with H_∞ and the (D)'s for logarithmic curves and a period for D_i equal to the corresponding period of the abelian integral (23) for any point on this curve.*

38. We now propose to form an abelian integral belonging to H_y of type

$$\int \frac{A(x, z)dx}{g_i(x, y)F_z'}$$

(A indeterminate polynomial adjoint to the curve), with no other discontinuities at finite distance than the points on D_i the corresponding logarithmic periods being all equal to $+1$. The relations which this imposes upon the coefficients of A, assumed of conveniently high degree are certainly compatible and moreover rational with respect to y, hence the integral is of the form

$$\int \frac{A(x, y, z)dx}{\psi(y)g_i(x, y)F_z'} = -\int V_i'dx,$$

where A is now an adjoint polynomial vanishing on the curve other than D_i projected on $g_i(x, y) = 0$. The abelian integral

$$-\frac{\partial}{\partial y}\int V_i'dx$$

behaves like a rational function at finite distance, hence as above there exists a rational function U_i' such that

$$\frac{P_i'(x, y, z)}{\chi(y)F_z'} = \frac{\partial U_i'}{\partial x} + \frac{\partial V_i'}{\partial y}$$

(P_i' adjoint polynomial, χ polynomial in y) this identity itself showing that U_i' is of the same type as V_i'.

For the sequel it is necessary to operate a further reduction so as to remove $\chi(y)$ concerning whose roots nothing is known. From the discussion of No. 27 it follows that we can find two functions U_i'', V_i'' of type

$$\frac{B(x, y, z)}{\theta(x)\theta_1(y)F_z'}, \qquad (\theta,\ \theta_1,\ B, \text{ polynomials, the last adjoint})$$

such that

$$\frac{P_i'(x, y, z)}{\chi(y)F_z'} - \frac{\partial U_i''}{\partial x} - \frac{\partial V_i''}{\partial y} = \frac{P_i(x, y, z)}{F_z'} = \frac{\partial U_i}{\partial x} + \frac{\partial V_i}{\partial y};$$

$$(U_i = U_i' - U_i'',\ V_i = V_i' - V_i'').$$

The abelian integral

$$-\int V_i dx \tag{26}$$

may have logarithmic periods with respect to points on curves H_x but to the points on a given H_{x_k} will correspond the same constant period say $2\pi i\beta_k$. The functions

$$V_i, \qquad V_i - \sum \frac{\beta_k}{x - x_k},$$

have the same partial derivatives with respect to y, hence the first may be replaced by the second in the identity which defines it, and the corresponding integral (26) will then have nowhere logarithmic periods with respect to points on the curves H_{x_i}. We will thus be able to write

$$\frac{P_i(x, y, z)}{F_z'} = \frac{\partial U_i}{\partial x} + \frac{\partial V_i}{\partial y},$$

the integral (26) having now no other logarithmic singularities at finite distance than the points on D_i with corresponding periods equal to $+1$.

39. Let us now return to the consideration of improper integrals of form (19). We have

$$\int\int\left[\frac{Q(x, y, z)}{\varphi(y)F_z'} - \sum \mu_i \frac{P_i(x, y, z)}{F_z'}\right] dxdy = \int\int\left(\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y}\right) dxdy.$$

The double integral at the right is again of form (19) but this time

$$-\int V dx$$

has no logarithmic singularities at finite distance and therefore the double integral has no periods. We conclude from this that the number $\rho - 1$ of distinct improper integrals of form (19) admitting of no linear combination without periods is the same as when only integrals

$$J_i = \int\int \frac{P_i(x, y, z)dxdy}{F_z'} = \int\int\left(\frac{\partial U_i}{\partial x} + \frac{\partial V_i}{\partial y}\right) dxdy$$

are considered. But if $\sum_{i=1}^{\rho'} \mu_i J_i$ is without periods there exists an integral of total differentials with H_∞, D_1, D_2, $\cdots$, $D_{\rho'}$, for sole logarithmic curves, and a period μ_i with respect to D_i and conversely. Hence the number $\rho - 1$ of distinct (J)'s of which no linear combination is without periods is equal to the maximum number of curves of F which together with H_∞ may not form the logarithmic curves of an integral of total differentials.

Let D_1, D_2, $\cdots$, $D_{\rho+1}$ be any $\rho + 1$ curves of F. There are two integrals of total differentials, I_1, I_2 with the respective logarithmic curves H_∞, D_1, D_2, $\cdots$, D_ρ and H_∞, D_1, $\cdots$, $D_{\rho-1}$, $D_{\rho+1}$, and periods α_1, α_2 relatively to H_∞. One of the two integrals of total differentials $\alpha_2 I_1 - \alpha_1 I_2$, I_1 (if $\alpha_1 = 0$) has D_1, D_2, $\cdots$, $D_{\rho+1}$ for sole logarithmic curves. Therefore *the number ρ is the maximum number of algebraic curves which may not be logarithmic curves of an integral of total differentials*, which is the very interpretation given for this number by Picard.

40. Let us recall briefly how Picard arrived at the number ρ.* Starting with an algebraic curve D_i he first formed an abelian integral of the third

* Picard-Simart, vol. II, p. 231, Ch. 9.

kind belonging to H_y

$$J_i = \int V_i dx$$

having the points on D_i for sole logarithmic points the corresponding periods being all $+1$. Let $\omega_{i,1}, \omega_{i,2}, \cdots, \omega_{i,2p+m-1}$, be a fundamental system of periods of J_i. When y turns around one of the critical points $\omega_{i\mu}$ becomes

$$\omega'_{i\mu} = \sum m_{\mu\nu}\omega_{i\nu} + m_\mu,$$

the integers $m_{\mu\nu}$, m_μ being independent of J_i. From this readily follows that by taking a sufficiently high number of these integrals we can form a linear combination with coefficients $b_i(y)$ rational in y, as

$$J = \sum b_i(y)J_i = \int V dx$$

with all periods constant. Hence as in No. 35 there exists an integral of total differentials

$$\int U dy + V dx,$$

with the (D)'s and the curve at infinity for sole logarithmic curves. This shows that there is a maximum ρ of the number of curves which may not become logarithmic curves of an integral of total differentials which is precisely the definition found for ρ in No. 39.

§ 8. *Summary of further developments concerning the number ρ.*

41. Only second in importance to Picard's work must be set Severi's discovery of the relation between the preceding theory and the distribution of algebraic systems of curves on the surface.* The writer may also mention his own contributions which connect the Picard-Severi theory to the topology.† We propose to dwell rapidly on these various developments.

By definition a continuous complete system $\{C\}$ of curves of F is said to be the sum of systems $\{A\}$, $\{B\}$, if there is a C composed of an A and a B. The system is perfectly defined and we write

$$\{C\} = \{A + B\}, \qquad \text{or} \qquad C = A + B.$$

From this to consideration of systems $\{A - B\}$, or more generally $\sum \lambda_i C_i$ where the (λ)'s are integers is but a step. We write $\sum \lambda_i C_i = 0$ and say that the curves are *algebraically dependent* if there is a C such that

$$\{C + \sum \lambda_i C_i\} = \{C\}.$$

* Sulla base per la totalita delle curve . . . , Math. Ann., vol. 62 (1906), pp. 194–225. La base minima pour la totalité des courbes . . . , Ann. Ec. Norm., ser. 3, vol. 25 (1908). See also two very interesting Memoires by Poincaré, Ann. Ec. Norm. Sup., ser. 3, vol. 27 (1910), pp. 55–108. Berliner Sitzungsb., vol. 10 (1911), pp. 28–55, where results closely related to Severi's are obtained by means of Abel's theorem.

† Rendiconti dei Lincei, loc. cit.

Let $[AB]$, $[A^2]$, denote the number of points common to generic curves A, B, or to two curves of $\{A\}$, respectively. Severi has established that: *In order that* $\lambda A = \lambda B$ (λ some integer) *it is necessary and sufficient that*

$$[A^2] = [AB] = [B^2].$$

By means of this criterium he showed finally that: *The curves* $C_1, C_2, \cdots, C_n$, *are algebraically dependent if and only if they are the logarithmic curves of an integral of total differentials of the third kind.* This is his fundamental result. From this follows that there can be found ρ independent curves $C_1, C_2, \cdots, C_\rho$ such that for any other C there is a relation

$$\lambda C = \sum_{i=1}^{\rho} \lambda_i C_i.$$

The curves C_i form a so-called *base* for all the curves of the surface. Thus on a ruled surface $\rho = 2$ a base being constituted by a generator and a plane section, on a quadric again $\rho = 2$ and two generators not of the same system form a base.

42. We have already mentioned that an algebraic curve forms a two-dimensional cycle. We will call *algebraic* cycle a cycle homologous to one of this nature. It may be shown, as is only partly done in the writer's Lincei note, that *a cycle is algebraic if and only if double integrals of the first kind have no periods with respect to it.* Hence ρ *is the number of distinct cycles with respect to which these integrals have no periods.* From this may be deduced the following fundamental result: *To a homology between algebraic cycles corresponds a relation of dependence between the curves and vice versa.* These theorems find numerous applications of which we may mention one—the first correct proof of a theorem stated by Nöther according to which the most general surface of order > 3 possesses no other curves than complete intersections.

THE POTENTIAL OF RING-SHAPED DISCS.

By E. P. Adams.

The method of obtaining solutions of Laplace's equation in cases where there is symmetry about an axis by means of zonal harmonics is well known. In brief, the method consists in expanding the known value of the potential on the axis in a power series in x, the distance measured along the axis, and in this series replacing x^n and $1/x^{n+1}$ by $r^n P_n$ (cos θ) and $(1/r^{n+1})P_n$ (cos θ) respectively. r is the distance in any direction, θ, measured from the origin. Each term in the result being a solution of Laplace's equation which reduces to the given value on the axis, it follows that the expression so obtained is the value of the potential at any point where the series is convergent. There are cases in which this method fails. We can, however, express the potential at any point due to an elementary ring of our distribution of matter by this method, and then in a number of important cases this expression can be integrated over the whole distribution in such a way that the resulting series is absolutely convergent for a definite region. In cases where both methods can be used some interesting consequences follow from a comparison of the results.

In the following we shall be concerned with thin circular ring-shaped discs of both single and double layers; we shall assume that the density in the former case and the strength in the latter case is a function of the distance from the center of the disc only. The internal and external radii of the disc are a and b respectively; ρ is the variable radius measured in the disc; and r, θ, the polar coördinates of any point in space measured with respect to the x axis which is supposed drawn normal to the plane of the disc from its center. As we deal only with cases of axial symmetry the second angular coördinate of any point in space is not required.

I. Potential due to single layer distributions.

1. Denoting the density of the distribution by $f(\rho)$ we have for the potential at any point on the axis:

$$V = 2\pi \int_a^b \frac{f(\rho)\rho d\rho}{\sqrt{x^2 + \rho^2}}. \tag{1}$$

We have to consider three regions:

1. $0 \leqq r \leqq a$,

2. $a \leqq r \leqq b$,

3. $b \leqq r$.

By first expressing the potential due to an elementary ring at any point within these three regions by the proper expansion in terms of zonal harmonics, and then integrating over the disc, we get:

$0 \leqq r \leqq a$

$$V = 2\pi \int_a^b f(\rho) \left\{ 1 - \frac{1}{2}\frac{r^2}{\rho^2}P_2 + \frac{1\cdot 3}{2\cdot 4}\frac{r^4}{\rho^4}P_4 - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{r^6}{\rho^6}P_6 + \cdots \right\} d\rho, \tag{2}$$

$r \geqq b$

$$V = 2\pi \int_a^b f(\rho) \left\{ \frac{\rho}{r} - \frac{1}{2}\frac{\rho^3}{r^3}P_2 + \frac{1\cdot 3}{2\cdot 4}\frac{\rho^5}{r^5}P_4 - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{\rho^7}{r^7}P_6 + \cdots \right\} d\rho, \tag{3}$$

$a \leqq r \leqq b$

$$V = 2\pi \int_a^r f(\rho) \left\{ \frac{\rho}{r} - \frac{1}{2}\frac{\rho^3}{r^3}P_2 + \frac{1\cdot 3}{2\cdot 4}\frac{\rho^5}{r^5}P_4 - \cdots \right\} d\rho$$

$$+ 2\pi \int_r^b f(\rho) \left\{ 1 - \frac{1}{2}\frac{r^2}{\rho^2}P_2 + \frac{1\cdot 3}{2\cdot 4}\frac{r^4}{\rho^4}P_4 - \cdots \right\} d\rho. \tag{4}$$

In these expressions P_n is written for $P_n(\cos\theta)$.

In any case, therefore, in which we can integrate $\int \rho^n f(\rho) d\rho$, where n is any positive or negative integer we can solve the problem completely.

Now we know that the resulting expression for the potential in any one of the three regions must be a solution of Laplace's equation:

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial V}{\partial\theta}\right) = 0. \tag{5}$$

Terms of the form $A_n r^n P_n$ and $(B_n/r^{n+1})P_n$ are known to be solutions of this equation. Therefore the sum of all the other terms must be a solution; in this way we are able to get some interesting special solutions. Our results must be continuous at the boundaries of the separate regions. The solution in the region $a \leqq r \leqq b$ must satisfy the relation at $x = 0$, $\theta = \pi/2$:

$$-\frac{\partial V}{\partial x} = 2\pi f(\rho). \tag{6}$$

Potential due to single layer distributions whose density is proportional to a positive or negative integral power of the radius.

2. Before giving the solution of this case in general it will be worth while to consider two special cases. First, suppose the disc to be of constant density, k. Applying (2), (3), and (4) we find:

$0 \leq r \leq a$

$$V = 2\pi k\left\{(b-a) + \frac{1}{2}\left(\frac{1}{b}-\frac{1}{a}\right)r^2P_2 - \frac{1\cdot 1}{2\cdot 4}\left(\frac{1}{b^3}-\frac{1}{a^3}\right)r^4P_4 + \cdots\right\}, \quad (7)$$

$r \geq b$

$$V = 2\pi k\left\{\frac{1}{2}\frac{b^2-a^2}{r} - \frac{1\cdot 1}{2\cdot 4}\frac{b^4-a^4}{r^3}P_2 + \frac{1\cdot 1\cdot 3}{2\cdot 4\cdot 6}\frac{b^6-a^6}{r^5}P_4 - \cdots\right\}, \quad (8)$$

$a \leq r \leq b$

$$\begin{aligned} V = 2\pi k\Big\{ & b\left(1 + \frac{1}{2}\frac{r^2}{b^2}P_2 - \frac{1\cdot 1}{2\cdot 4}\frac{r^4}{b^4}P_4 + \frac{1\cdot 1\cdot 3}{2\cdot 4\cdot 6}\frac{r^6}{b^6}P_6 - \cdots\right) \\ & - a\left(\frac{1}{2}\frac{a}{r} - \frac{1\cdot 1}{2\cdot 4}\frac{a^3}{r^3}P_2 + \frac{1\cdot 1\cdot 3}{2\cdot 4\cdot 6}\frac{a^5}{r^5}P_4 - \cdots\right) \\ & - r\left(\frac{1}{2} + \frac{1}{2}\left(1+\frac{1}{4}\right)P_2 - \frac{1\cdot 3}{2\cdot 4}\left(\frac{1}{3}+\frac{1}{6}\right)P_4 \right. \\ & \left. + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\left(\frac{1}{5}+\frac{1}{8}\right)P_6 - \cdots\right)\Big\}. \end{aligned} \quad (9)$$

Putting $r = a$ in (7) and (9) we see that they both give the same result; and putting $r = b$ in (8) and (9) we get the same result. Thus the conditions of continuity are satisfied. The last term in (9) must be a solution of Laplace's equation; we should therefore expect this term to be $rP_1(\cos\theta)$. This is the value we should get for it if we had solved this problem by integrating (1) first and then expanding it in terms of zonal harmonics in the usual way. Further, with this value the relation (6) is satisfied. We therefore have:

$$P_1 = \cos\theta = \frac{1}{2} + \frac{1}{2}\left(1+\frac{1}{4}\right)P_2 - \frac{1\cdot 3}{2\cdot 4}\left(\frac{1}{3}+\frac{1}{6}\right)P_4 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\left(\frac{1}{5}+\frac{1}{8}\right)P_6 - \cdots. \quad (10)$$

The infinite series in (10) is absolutely convergent in the interval $-\pi/2 \leq \theta \leq +\pi/2$.

3. We next consider the case where the density is inversely proportional to the distance from the center of the disc. According to the usual method we should have to develop (1) in a series, with $f(\rho) = k\rho^{-1}$, after integrating. On the axis we find:

$$V = 2\pi k \log\frac{b + \sqrt{b^2+x^2}}{a + \sqrt{a^2+x^2}}.$$

There is no difficulty here provided that $x > b$ or $x < a$. But in the region $b > x > a$, we find:

$$V = 2\pi k \left\{ \log \frac{2b}{x} + \frac{1\cdot 1}{2\cdot 2}\frac{x^2}{b^2} - \frac{1\cdot 1\cdot 3}{2\cdot 4\cdot 4}\frac{x^4}{b^4} + \cdots \right.$$

$$\left. - \frac{a}{x} + \frac{1\cdot 1}{2\cdot 3}\frac{a^3}{x^3} - \frac{1\cdot 1\cdot 3}{2\cdot 4\cdot 5}\frac{a^5}{x^5} + \cdots \right\}. \tag{11}$$

Here it is the term $\log 2b/x$ that causes difficulty. It might be thought that we could use the zonal harmonics of the second kind, but these become infinite on the axis and so are useless for these problems. If we use (4) we get for the potential at any point in the region $a < r < b$:

$$V = 2\pi k \left\{ \log \frac{b}{r} + 1 - \frac{1}{2}\left(\frac{1}{2} + \frac{1}{3}\right) P_2 + \frac{1\cdot 3}{2\cdot 4}\left(\frac{1}{4} + \frac{1}{5}\right) P_4 - \cdots \right.$$

$$+ \frac{1\cdot 1}{2\cdot 2}\frac{r^2}{b^2} P_2 - \frac{1\cdot 1\cdot 3}{2\cdot 4\cdot 4}\frac{r^4}{b^4} P_4 + \frac{1\cdot 1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 6}\frac{r^6}{b^6} P_6 - \cdots \tag{12}$$

$$\left. - \frac{a}{r} + \frac{1\cdot 1}{2\cdot 3}\frac{a^3}{r^3} P_2 - \frac{1\cdot 1\cdot 3}{2\cdot 4\cdot 5}\frac{a^5}{r^5} P_4 + \cdots \right\}.$$

Comparing (11) and (12) we see that $\log 2b/x$ on the axis expands into $\log (b/r) + L_0 (\cos \theta)$, where

$$L_0 = 1 - \frac{1}{2}\left(\frac{1}{2} + \frac{1}{3}\right) P_2 + \frac{1\cdot 3}{2\cdot 4}\left(\frac{1}{4} + \frac{1}{5}\right) P_4 - \cdots. \tag{13}$$

The other terms in (12) are formed from the corresponding terms in (11) in the usual way. Now from the fact that $L_0 - \log r$ must be a solution of (5), we find that L_0 satisfies the equation:

$$\frac{\partial}{\partial \theta}\left(\sin \theta \frac{\partial L}{\partial \theta} \right) = \sin \theta$$

whose complete solution is:

$$L_0 = c_1 \log \tan \frac{\theta}{2} - \log \sin \theta + c_2.$$

Since L_0 must remain finite on the axis we must have $c_1 = 1$. Further, we see that we must have:

$$\log 2 = 1 - \frac{1}{2}\left(\frac{1}{2} + \frac{1}{3}\right) + \frac{1\cdot 3}{2\cdot 4}\left(\frac{1}{4} + \frac{1}{5}\right) - \cdots \tag{14}$$

and that:

$$L_0 = \log \frac{4}{1 + \cos \theta} \tag{15}$$

$\log r(1 + \cos \theta)$ is thus a solution of Laplace's equation, which is easily verified. It may be shown without difficulty that the conditions of continuity are satisfied, as well as the relation (6).

4. In considering the general case where the density varies as any integral power of the radius it is necessary to take separately even and odd, and positive and negative powers. Let us consider first the case where the density is equal to $k\rho^{2n}$. On the axis we have:

$$V = 2\pi k \int_a^b \frac{\rho^{2n+1}d\rho}{\sqrt{x^2+\rho^2}}.$$

This can be integrated, giving:

$$V = \frac{2\pi k}{2n+1}\left[\sqrt{x^2+\rho^2}\left(\rho^{2n} - \frac{2nx^2\rho^{2n-2}}{(2n-1)} + \frac{2n(2n-2)x^4\rho^{2n-4}}{(2n-1)(2n-3)} - \cdots + (-1)^n\frac{2n(2n-2)(2n-4)\cdots 2}{(2n-1)(2n-3)\cdots 1}x^{2n}\right)\right]_a^b.$$

In this expression, after putting in the limits, $\sqrt{x^2+b^2}$ is to be developed by the binomial theorem in an infinite series in x/b, and $\sqrt{x^2+a^2}$ in an infinite series in a/x. After multiplying the finite sum by the two infinite series, x^n is to be replaced by $r^nP_n(\cos\theta)$ and x^{-n} by $(1/r^n)P_{n-1}(\cos\theta)$ and the result will be the solution of the problem in the region $a \leq r \leq b$. This problem can also be solved by applying (4) to this case, putting $f(\rho) = k\rho^{2n}$. On comparing the results of these two solutions we find that we must have:

$$\frac{2\cdot 4\cdot 6\cdots 2n}{3\cdot 5\cdot 7\cdots(2n+1)}P_{2n+1} = \sum_{m=0}^{\infty}(-1)^{m+n+1}\frac{1\cdot 3\cdot 5\cdots 2m-1}{2\cdot 4\cdot 6\cdots 2m}\left(\frac{1}{2m-2n-1} + \frac{1}{2m+2n+2}\right)P_{2m}. \tag{16}$$

We are thus able to express any zonal harmonic of odd integral order as an absolutely convergent infinite series, valid between $\theta = -\pi/2$ and $\theta = +\pi/2$, in all the zonal harmonics of even integral order. Equation (10) is seen to be a special case of this general formula for $n = 0$. We get the same result by taking the density to be equal to $k\rho^{-2n-3}$.

5. When we consider simple layers whose density is inversely proportional to an even power or directly proportional to an odd power of the distance from the center we get results of an entirely different character. Assuming the density to be equal to k/ρ^{2n+2}, we find by the methods already employed:

$$\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots 2n}\log 2 = \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots 2n}\left(\frac{1}{1\cdot 2}+\frac{1}{3\cdot 4}+\cdots\right.$$
$$\left.+\frac{1}{(2n-1)2n}\right)+\sum_{m=0}^{\infty}(-1)^{m+n}\frac{1\cdot 3\cdot 5\cdots(2m-1)}{2\cdot 4\cdot 6\cdots 2m}\left(\frac{1}{2m-2n}\right. \tag{17}$$
$$\left.+\frac{1}{2m+2n+1}\right).$$

In this expression n may be any positive integer including 0. It should be noted that the first term in the parenthesis in the infinite series becomes infinite for the particular value of $m = n$. This term is to be ignored for that value of m. This same result is obtained when the density is taken to be equal to $k\rho^{2n-1}$. With $n = 0$ we get (14) as a special case.

With the density proportional to ρ^{2n+2}, this method leads to a particular solution of Laplace's equation:

$$V = \frac{1}{r^{2n+1}}\left(\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots 2n}\log r\cdot P_{2n} + L_{2n}\right). \tag{18}$$

Taking the density proportional to ρ^{2n-1} we find the corresponding solution of Laplace's equation:

$$V = r^{2n}\left(\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots 2n}\log r\cdot P_{2n} - L_{2n}\right). \tag{19}$$

In these expressions L_{2n} is the absolutely convergent series:

$$L_{2n} = \sum_{m=0}^{\infty}(-1)^{m+n}\frac{1\cdot 3\cdot 5\cdots(2m-1)}{2\cdot 4\cdot 6\cdots 2m}\left(\frac{1}{2m-2n}+\frac{1}{2m+2n+1}\right)P_{2m}. \tag{20}$$

By substituting either (18) or (19) in Laplace's equation (5) we find that L_{2n} satisfies the equation:

$$\frac{\partial^2 y}{\partial\theta^2}+\cot\theta\frac{\partial y}{\partial\theta}+2n(2n+1)y = (4n+1)\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots 2n}P_{2n}. \tag{21}$$

6. As we shall meet similar equations in some of the other applications of this method let us write it, putting $\cos\theta = \mu$, and $2n = m$,

$$(1-\mu^2)\frac{\partial^2 y}{\partial\mu^2}-2\mu\frac{\partial y}{\partial\mu}+m(m+1)y = AP_m(\mu), \tag{22}$$

where A is a numerical constant. This equation with the right-hand member equal to 0 is Legendre's equation whose complete solution we know. Equation (22) can therefore be solved by the method of variation of parameters; we find for its solution:

$$y = c_1 Q_m + c_2 P_m + A Q_m \int P_m^2 d\mu - A P_m \int P_m Q_m d\mu,$$

where Q_m is the zonal harmonic of order m of the second kind. But as we are concerned only with solutions which remain finite on the axis one of the integration constants must be determined so as to satisfy this condition. This leads to a solution of the form:

$$y = a_m P_m \cdot \log (1 + \mu) + G + c_2 P_m,$$

where G is a polynomial of degree $m - 1$. Since any polynomial can be expressed as a finite sum of zonal harmonics of decreasing order beginning with that one of order equal to the degree of the polynomial, we can get our solution quickly by assuming:

$$y = a_m P_m \log (1 + \mu) + c_2 P_m + \sum_{k=1}^{m} a_{m-k} P_{m-k}. \tag{23}$$

When this is substituted in (22) we find:

$$\begin{aligned} a_m &= - \frac{A}{2m + 1}, \\ a_{m-k} &= - (-1)^k \frac{2A}{2m + 1} \frac{2m - 2k + 1}{k(2m - k + 1)}. \end{aligned} \tag{24}$$

The one remaining constant, c_2, may be found from the condition for $\theta = 0$. The value of this constant for the solution of (21) we find from (17), (20) and (23):

$$c_2 = \frac{1 \cdot 3 \cdot 5 \cdots (2n - 1)}{2 \cdot 4 \cdot 6 \cdots 2n} 2 \log 2 - \sum_{k=1}^{2n} a_{2n-k}$$
$$- \frac{1 \cdot 3 \cdot 5 \cdots (2n - 1)}{2 \cdot 4 \cdot 6 \cdots 2n} \left(\frac{1}{1 \cdot 2} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{(2n - 1)2n} \right).$$

7. Although the infinite series represented by (16) and (20) are absolutely convergent for $-\pi/2 \leq \theta \leq +\pi/2$, their derivatives are not convergent. Now it is the infinite series that we get when we apply (4) to determine the value of the potential at any point in the region $a \leq r \leq b$. In order that (6) may be satisfied it is necessary to substitute for the infinite series their equivalents given by (21) and (24). Having done this all the conditions of the problem will be satisfied.

Potential due to single layer distributions whose density is a logarithmic function of the radius.

8. If we take the density of the single layer distribution to be proportional to $\log \rho/\rho^{2n+3}$, where n is any positive integer including 0, it is possible to integrate the potential on the axis and then expand it in a

series which will be convergent in the region $a \leqq r \leqq b$. We can also apply (4) to this case directly. On comparing the results obtained by the two methods we find the following:

If we define the infinite series:

$$M_{2n+1} = \sum_{m=0}^{\infty} (-1)^{n+m} \frac{1\cdot3\cdot5\cdots(2m-1)}{2\cdot4\cdot6\cdots2m} \left\{ \frac{1}{(2n-2m+1)^2} - \frac{1}{(2n+2m+2)^2} \right\} P_{2m} \quad (25)$$

we have a solution of Laplace's equation:

$$V = \frac{1}{r^{2n+2}} \left\{ \frac{2\cdot4\cdot6\cdots2n}{1\cdot3\cdot5\cdots(2n+1)} \log r \cdot P_{2n+1} + M_{2n+1} \right\} \quad (26)$$

and M_{2n+1} satisfies the equation:

$$\frac{\partial^2 y}{\partial \theta^2} + \cot\theta \frac{\partial y}{\partial \theta} + (2n+2)(2n+1)y = (4n+3)\frac{2\cdot4\cdot6\cdots2n}{1\cdot3\cdot5\cdots(2n+1)} P_{2n+1}.$$

Putting $\cos\theta = \mu$, and $2n + 1 = m$, we get equation (22) whose solution we have found subject to the condition of remaining finite on the axis. We find in addition that the following relation must hold:

$$\frac{2\cdot4\cdot6\cdots2n}{1\cdot3\cdot5\cdots(2n+1)} \log 2 = \frac{2\cdot4\cdot6\cdots2n}{1\cdot3\cdot5\cdots(2n+1)} \left\{ \frac{1}{2n+1} + \frac{1}{2n(2n-1)} + \frac{1}{(2n-2)(2n-3)} + \cdots + \frac{1}{2\cdot1} \right\}$$
$$- \sum_{m=0}^{\infty} (-1)^{n+m} \frac{1\cdot3\cdot5\cdots2m-1}{2\cdot4\cdot6\cdots2m} \left\{ \frac{1}{(2n-2m+1)^2} - \frac{1}{(2n+2m+2)^2} \right\}.$$

Particular cases of this last formula for $n = 0$ and $n = 1$ are:

$$\log 2 = 1 - \left(1 - \frac{1}{2^2}\right) + \frac{1}{2}\left(1 - \frac{1}{4^2}\right) - \frac{1\cdot3}{2\cdot4}\left(\frac{1}{3^2} - \frac{1}{6^2}\right) + \frac{1\cdot3\cdot5}{2\cdot4\cdot6}\left(\frac{1}{5^2} - \frac{1}{8^2}\right) - \cdots,$$

$$\frac{2}{3}\log 2 = \frac{5}{9} + \left(\frac{1}{3^2} - \frac{1}{4^2}\right) - \frac{1}{2}\left(1 - \frac{1}{6^2}\right) + \frac{1\cdot3}{2\cdot4}\left(1 - \frac{1}{8^2}\right) - \frac{1\cdot3\cdot5}{2\cdot4\cdot6}\left(\frac{1}{3^2} - \frac{1}{10^2}\right) + \frac{1\cdot3\cdot5\cdot7}{2\cdot4\cdot6\cdot8}\left(\frac{1}{5^2} - \frac{1}{12^2}\right) - \cdots.$$

9. If we take the density of the single layer distribution to be proportional to $\rho^{2n} \log \rho$, where n is any positive integer, including 0, we

get the same results with the exception of the particular solution of Laplace's equation. Corresponding to (26) we now find a solution:

$$V = \frac{2\cdot 4\cdot 6\cdots 2n}{1\cdot 3\cdot 5\cdots(2n+1)} r^{2n+1}\log r\cdot P_{2n+1} - r^{2n+1}M_{2n+1},$$

where M_{2n+1} is the infinite series (25).

10. In all the problems considered so far it has been possible to integrate the value of the potential on the axis in terms of the elementary functions, and our results have been obtained by comparing the expansion of this expression with the one arising from the application of equation (4). But there are many cases in which this cannot be done, and it is of some interest to see what results we get from the use of (4) alone.

Assume the density of the single layer to be $k(\log \rho/\rho)$. Using (4) we find for the potential in the region $a \leqq r \leqq b$:

$$\begin{aligned} V = 2\pi k\Big\{ &\log r\left[1 - \frac{1}{2}\left(\frac{1}{2}+\frac{1}{3}\right)P_2 + \frac{1\cdot 3}{2\cdot 4}\left(\frac{1}{4}+\frac{1}{5}\right)P_4 - \cdots\right] \\ &- \left[1 + \frac{1}{2}\left(\frac{1}{2^2}-\frac{1}{3^2}\right)P_2 - \frac{1\cdot 3}{2\cdot 4}\left(\frac{1}{4^2}-\frac{1}{5^2}\right)P_4 + \cdots\right] \\ &- \frac{1}{2}(\log r)^2 + \frac{1}{2}(\log b)^2 \\ &- \frac{a}{r}(\log a - 1) + \frac{1}{2}\frac{a^3}{r^3}\left(\frac{\log a}{3} - \frac{1}{3^2}\right)P_2 - \frac{1\cdot 3}{2\cdot 4}\frac{a^5}{r^5}\left(\frac{\log a}{5} - \frac{1}{5^2}\right)P_4 + \cdots \\ &+ \frac{1}{2}\frac{r^2}{b^2}\left(\frac{\log b}{2} + \frac{1}{2^2}\right)P_2 - \frac{1\cdot 3}{2\cdot 4}\frac{r^4}{b^4}\left(\frac{\log b}{4} + \frac{1}{4^2}\right)P_4 + \cdots\Big\}. \end{aligned}$$

The coefficient of $\log r$ we have already had; it is the L_0 of (13) whose value we have found to be:

$$L_0 = \log\frac{4}{1+\cos\theta}.$$

Using this value we find that (6) is satisfied by this solution. It is now of interest to try to determine the value of the other infinite series of zonal harmonics which enters into this solution. Writing this:

$$f_0 = 1 + \frac{1}{2}\left(\frac{1}{2^2}-\frac{1}{3^2}\right)P_2 - \frac{1\cdot 3}{2\cdot 4}\left(\frac{1}{4^2}-\frac{1}{5^2}\right)P_4 + \cdots \tag{27}$$

we see that:

$$V = \log r\cdot\log\frac{4}{1+\cos\theta} - f_0 - \frac{1}{2}(\log r)^2$$

is a solution of Laplace's equation since all the other terms are separately solutions. We thus find that f_0 satisfies the equation

$$\frac{\partial^2 f_0}{\partial \theta^2} + \cot\theta \frac{\partial f_0}{\partial \theta} + 1 = \log \frac{4}{1 + \cos\theta}.$$

This equation cannot be solved in terms of the elementary functions. By the usual methods we find a solution with one arbitrary constant determined so that f_0 will remain finite on the axis:

$$f_0 = c_2 - \log 2 \log (1 + \cos\theta) - \left(\frac{1 - \cos\theta}{2}\right) - \frac{1}{2^2}\left(\frac{1 - \cos\theta}{2}\right)^2 - \frac{1}{3^2}\left(\frac{1 - \cos\theta}{2}\right)^3 - \cdots.$$

When $\theta = 0$,

$$f_0 = c_2 - (\log 2)^2.$$

From (27) for $\theta = 0$, we find, accurate to four places:

$$f_0 = 1.0627\cdots.$$

Hence

$$c_2 = 1.5432\cdots.$$

For $\theta = \pi/2$, we have:

$$f_0 = c_2 - \frac{1}{2} - \frac{1}{2^2 2^2} - \frac{1}{3^2 2^3} - \frac{1}{4^2 2^4} - \cdots.$$

Using the known series:

$$\sum_{m=1}^{\infty} \frac{1}{2^m m^2} = \frac{\pi^2}{12} - \frac{1}{2}(\log 2)^2$$

we find:

$$0.9609\cdots = 1 - \left(\frac{1}{2}\right)^2\left(\frac{1}{2^2} - \frac{1}{3^2}\right) - \left(\frac{1\cdot 3}{2\cdot 4}\right)^2\left(\frac{1}{4^2} - \frac{1}{5^2}\right) - \cdots.$$

II. Potential due to double layer distributions.

11. The potential at any point due to a double layer distribution of unit strength is equal to the solid angle subtended by the distribution at the point. We shall again consider a circular ring-shaped disc of internal and external radii a and b, whose strength, $f(\rho)$ is a function of the distance from the center only.

The solid angle subtended by an elementary ring of radius ρ and width $d\rho$ at a point, x, on the axis is

$$d\omega = 2\pi \frac{x\rho d\rho}{(x^2 + \rho^2)^{3/2}}.$$

We accordingly get for the potential at any point on the axis:

$$V = 2\pi x \int_a^b f(\rho) \frac{\rho d\rho}{(x^2 + \rho^2)^{3/2}}. \tag{28}$$

By developing the potential due to an elementary ring of the distribution in a series of zonal harmonics, and then integrating over the disc, we get in the three regions:

$r < a$

$$V = 2\pi \int_a^b f(\rho) \left\{ \frac{r}{\rho^2} P_1 - \frac{3}{2} \frac{r^3}{\rho^4} P_3 + \frac{3 \cdot 5}{2 \cdot 4} \frac{r^5}{\rho^6} P_5 - \cdots \right\} d\rho, \tag{29}$$

$r > b$

$$V = 2\pi \int_a^b f(\rho) \left\{ \frac{\rho}{r^2} P_1 - \frac{3}{2} \frac{\rho^3}{r^4} P_3 + \frac{3 \cdot 5}{2 \cdot 4} \frac{\rho^5}{r^6} P_5 - \cdots \right\} d\rho, \tag{30}$$

$a < r < b$

$$\begin{aligned} V = 2\pi \int_a^r f(\rho) \left\{ \frac{\rho}{r^2} P_1 - \frac{3}{2} \frac{\rho^3}{r^4} P_3 + \frac{3 \cdot 5}{2 \cdot 4} \frac{\rho^5}{r^6} P_5 - \cdots \right\} d\rho \\ + 2\pi \int_r^b f(\rho) \left\{ \frac{r}{\rho^2} P_1 - \frac{3}{2} \frac{r^3}{\rho^4} P_3 + \frac{3 \cdot 5}{2 \cdot 4} \frac{r^5}{\rho^6} P_5 - \cdots \right\} d\rho. \end{aligned} \tag{31}$$

As in the problem of single layer distributions V must be a solution of Laplace's equation (5); there must be continuity at $r = a$ between the results of (29) and (31), and between (30) and (31) at $r = b$. In place of relation (6) we must now have, at $x = 0$, $\theta = \pi/2$, $b > \rho > a$:

$$V = 2\pi f(\rho). \tag{32}$$

In order to obtain (31) we have developed $[1 + (\rho^2/x^2)]^{-3/2}$ and $[1 + (x^2/\rho^2)]^{-3/2}$ by the binomial theorem. In integrating over the disc it has been assumed that we could still use these series when $\rho = x$. Although this procedure cannot be justified we shall see that we can get correct results by use of it.

We shall now consider a few special cases. As the simplest application of a double layer distribution is that of a magnetic shell we shall use the term "magnetic shell" in speaking of such distributions.

12. Consider first the case of a uniform circular magnetic shell. This is the simplest case whose solution gives us that of a uniform electric current flowing in a thin wire coinciding with the contour of the shell. The value of the potential, obtained by integrating (28) with $a = 0$, $f(\rho) = k$, expanding in a series and introducing zonal harmonics in the usual way, is:

$$V = 2\pi k \left\{ 1 - \frac{r}{b} P_1 + \frac{1}{2} \frac{r^3}{b^3} P_3 - \frac{1 \cdot 3}{2 \cdot 4} \frac{r^5}{b^5} P_5 + \cdots \right\} r < b. \tag{33}$$

If we now apply (31) to this case we get:

$$V = 2\pi k\left\{1 - \frac{r}{b}P_1 + \frac{1}{2}\frac{r^3}{b^3}P_3 - \frac{1\cdot 3}{2\cdot 4}\frac{r^5}{b^5}P_5 + \cdots\right\}$$

$$- 2\pi k\left\{1 - \left(1 + \frac{1}{2}\right)P_1 + \frac{3}{2}\left(\frac{1}{3} + \frac{1}{4}\right)P_3 \right. \quad (34)$$

$$\left. - \frac{3\cdot 5}{2\cdot 4}\left(\frac{1}{5} + \frac{1}{6}\right)P_5 + \cdots\right\}.$$

Hence we must have:

$$1 = P_0 = \left(1 + \frac{1}{2}\right)P_1 - \frac{3}{2}\left(\frac{1}{3} + \frac{1}{4}\right)P_3 + \frac{3\cdot 5}{2\cdot 4}\left(\frac{1}{5} + \frac{1}{6}\right)P_5 - \cdots. \quad (35)$$

The infinite series in (35) is divergent. Its divergency arises from the expansion of $[1 + (x^2/\rho^2)]^{-3/2}$ by the binomial theorem and making use of this expansion in integrating for $x = \rho$. We see now that this method leads to correct results if we define the value of the divergent series (35) as unity. For $\theta = 0$, $P_n = 1$, and the series is obviously equal to unity. For $\theta = \pi/2$, $P_1 = P_2 = P_3 = \cdots = 0$, and the numerical value of the series is 0. For any value of θ less than $\pi/2$ the series appears to approach more nearly to 1 the more terms that are used.

13. We consider next a circular ring-shaped magnetic shell whose strength is directly proportional to the radius. This case is easily realized practically. If we have fine wire wound in the form of a ring-shaped disc, of radii a and b, and send a unit electric current through it, the equivalent magnetic shell is a compound one consisting of the superposition of (1) a uniform magnetic shell of radius b, strength kb; (2) a uniform magnetic shell of radius a, strength $-ka$; (3) a variable ring-shaped magnetic shell, radii a and b, strength $-k\rho$. If N is the total number of turns of wire, $k = N/(b - a)$. The potential due to the first two distributions we can write down at once from the results already obtained. We now proceed to find the potential due to the third distribution.

The potential on the axis is:

$$V = 2\pi k\int_a^b \frac{x\rho^2 d\rho}{(x^2 + \rho^2)^{3/2}},$$

in which $k\rho$ is taken as the strength of the shell. This integral can be evaluated and expanded, giving:

$$V = 2\pi k\left\{x\log\frac{2b}{x} - x + b\left(\frac{1\cdot 3}{2\cdot 2}\frac{x^3}{b^3} - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 4}\frac{x^5}{b^5} + \frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 6}\frac{x^7}{b^7} - \cdots\right)\right.$$

$$\left. - a\left(\frac{1}{3}\frac{a^2}{x^2} - \frac{1\cdot 3}{2\cdot 5}\frac{a^4}{x^4} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 7}\frac{a^6}{x^6} - \cdots\right)\right\}.$$

Using equation (31) we find:

$$V = 2\pi k \left\{ rP_1 \log \frac{b}{r} + r\left(\frac{1}{3}P_1 - \frac{3}{2}\left(\frac{1}{2}+\frac{1}{5}\right)P_3 + \frac{3\cdot 5}{2\cdot 4}\left(\frac{1}{4}+\frac{1}{7}\right)P_5 - \cdots\right)\right.$$
$$+ b\left(\frac{1\cdot 3}{2\cdot 2}\frac{r^3}{b^3}P_3 - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 4}\frac{r^5}{b^5}P_5 + \frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 6}\frac{r^7}{b^7}P_7 - \cdots\right)$$
$$\left. - a\left(\frac{1}{3}\frac{a^2}{r^2}P_1 - \frac{1\cdot 3}{2\cdot 5}\frac{a^4}{r^4}P_3 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 7}\frac{a^6}{r^6}P_5 - \cdots\right)\right\}.$$

If we write:

$$L_1 = \frac{1}{3}P_1 - \frac{3}{2}\left(\frac{1}{2}+\frac{1}{5}\right)P_3 + \frac{3\cdot 5}{2\cdot 4}\left(\frac{1}{4}+\frac{1}{7}\right)P_5 - \cdots \tag{36}$$

we see that

$$x \log \frac{2b}{x} - x$$

on the axis, expands into

$$rL_1 + rP_1 \log \frac{b}{r}.$$

Hence, putting $\theta = 0$, we find:

$$\log 2 = 1 + \frac{1}{3} - \frac{3}{2}\left(\frac{1}{2}+\frac{1}{5}\right) + \frac{3\cdot 5}{2\cdot 4}\left(\frac{1}{4}+\frac{1}{7}\right) - \cdots. \tag{37}$$

Further, we find that L_1 is a solution of:

$$\frac{\partial^2 y}{\partial\theta^2} + \cot\theta\frac{\partial y}{\partial\theta} + 2y = 3P_1.$$

The solution of this equation, to agree with the infinite series (36) is:

$$L_1 = 1 - 2\cos\theta + \cos\theta\log\frac{4}{1+\cos\theta}. \tag{38}$$

The divergent series (36) which enters into the solution of this problem must be replaced by (38) in order to get correct results.

We can now write the value of the magnetic potential due to the distribution of electric currents described above. The result is:

$$V = 2\pi k \left\{ b\left(1 - \frac{r}{b}P_1 - \frac{1\cdot 1}{2\cdot 2}\frac{r^3}{b^3}P_3 + \frac{1\cdot 1\cdot 3}{2\cdot 4\cdot 4}\frac{r^5}{b^5}P_5 - \cdots\right)\right.$$
$$- a\left(\frac{1\cdot 1}{2\cdot 3}\frac{a^2}{r^2}P_1 - \frac{1\cdot 1\cdot 3}{2\cdot 4\cdot 5}\frac{a^4}{r^4}P_3 + \frac{1\cdot 1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 7}\frac{a^6}{r^6}P_5 - \cdots\right)$$
$$\left. + r\left(P_1\log\frac{r(1+\cos\theta)}{4b} - 1 + 2P_1\right)\right\}\quad a < r < b.$$

We see that for $\theta = \pi/2$, $x = 0$, $r = \rho$:

$$V = 2\pi k\left\{1 - \frac{r}{b}P_1 + \frac{1}{2}\frac{r^3}{b^3}P_3 - \frac{1\cdot 3}{2\cdot 4}\frac{r^5}{b^5}P_5 + \cdots\right\}$$

$$- 2\pi k\left\{1 - \left(1 + \frac{1}{2}\right)P_1 + \frac{3}{2}\left(\frac{1}{3} + \frac{1}{4}\right)P_3 \right. \tag{34}$$

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in which $k\rho$ is taken as the strength of the shell. This integral can be evaluated and expanded, giving:

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Using equation (31) we find:

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$$+ b\left(\frac{1\cdot 3}{2\cdot 2}\frac{r^3}{b^3}P_3 - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 4}\frac{r^5}{b^5}P_5 + \frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 6}\frac{r^7}{b^7}P_7 - \cdots\right)$$
$$\left. - a\left(\frac{1}{3}\frac{a^2}{r^2}P_1 - \frac{1\cdot 3}{2\cdot 5}\frac{a^4}{r^4}P_3 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 7}\frac{a^6}{r^6}P_5 - \cdots\right)\right\}.$$

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$$L_1 = \frac{1}{3}P_1 - \frac{3}{2}\left(\frac{1}{2}+\frac{1}{5}\right)P_3 + \frac{3\cdot 5}{2\cdot 4}\left(\frac{1}{4}+\frac{1}{7}\right)P_5 - \cdots \tag{36}$$

we see that

$$x\log\frac{2b}{x} - x$$

on the axis, expands into

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Hence, putting $\theta = 0$, we find:

$$\log 2 = 1 + \frac{1}{3} - \frac{3}{2}\left(\frac{1}{2}+\frac{1}{5}\right) + \frac{3\cdot 5}{2\cdot 4}\left(\frac{1}{4}+\frac{1}{7}\right) - \cdots. \tag{37}$$

Further, we find that L_1 is a solution of:

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The solution of this equation, to agree with the infinite series (36) is:

$$L_1 = 1 - 2\cos\theta + \cos\theta\log\frac{4}{1+\cos\theta}. \tag{38}$$

The divergent series (36) which enters into the solution of this problem must be replaced by (38) in order to get correct results.

We can now write the value of the magnetic potential due to the distribution of electric currents described above. The result is:

$$V = 2\pi k \left\{ b\left(1 - \frac{r}{b}P_1 - \frac{1\cdot 1}{2\cdot 2}\frac{r^3}{b^3}P_3 + \frac{1\cdot 1\cdot 3}{2\cdot 4\cdot 4}\frac{r^5}{b^5}P_5 - \cdots\right)\right.$$
$$- a\left(\frac{1\cdot 1}{2\cdot 3}\frac{a^2}{r^2}P_1 - \frac{1\cdot 1\cdot 3}{2\cdot 4\cdot 5}\frac{a^4}{r^4}P_3 + \frac{1\cdot 1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 7}\frac{a^6}{r^6}P_5 - \cdots\right)$$
$$\left. + r\left(P_1\log\frac{r(1+\cos\theta)}{4b} - 1 + 2P_1\right)\right\} a < r < b.$$

We see that for $\theta = \pi/2$, $x = 0$, $r = \rho$:

$$V = 2\pi k(b - \rho)$$

showing that our solution satisfies (32).

14. As in the case of single layer distributions we can express the results, in a general form, of assuming the strength of the ring-shaped magnetic shell to be proportional to any integral positive or negative power of the radius.

If the strength of the shell varies directly as an even power or inversely as an odd power of the radius we find a divergent infinite series of zonal harmonics of odd order which must be taken as defining a zonal harmonic of even order:

$$\frac{2\cdot4\cdot6\cdots2n}{3\cdot5\cdot7\cdots(2n-1)}P_{2n}$$
$$= \sum_{m=0}^{\infty}(-1)^{m+n}\frac{3\cdot5\cdot7\cdots(2m+1)}{2\cdot4\cdot6\cdots2m}\left(\frac{1}{2m-2n+1}\right.$$
$$\left.+\frac{1}{2m+2n+2}\right)P_{2m+1}. \quad (39)$$

For $n = 0$ we get (35) as a special case of this general formula.

15. If the strength of the magnetic shell varies directly as an odd power or inversely as an even power of the radius we get the following results by comparing the value of the potential obtained by integrating its value on the axis, and expanding, with that obtained by the use of (31).

$$\frac{3\cdot5\cdot7\cdots2n+1}{2\cdot4\cdot6\cdots2n}\log 2 = \frac{3\cdot5\cdot7\cdots2n+1}{2\cdot4\cdot6\cdots2n}\left(\frac{1}{2n+1}+\frac{1}{1\cdot2}+\frac{1}{3\cdot4}\right.$$
$$\left.+\cdots+\frac{1}{(2n-1)2n}\right)$$
$$+\sum_{m=0}^{\infty}(-1)^{m+n}\frac{1\cdot3\cdot5\cdots2m+1}{2\cdot4\cdot6\cdots2m}\left(\frac{1}{2m-2n}+\frac{1}{2m+2n+3}\right). \quad (40)$$

(37) is the special case of this for $n = 0$. Further, if we write the infinite series:

$$L_{2n+1} = \sum_{m=0}^{\infty}(-1)^{m+n}\frac{3\cdot5\cdot7\cdots2m+1}{2\cdot4\cdot6\cdots2m}\left(\frac{1}{2m-2n}\right.$$
$$\left.+\frac{1}{2m+2n+3}\right)P_{2m+1} \quad (41)$$

we have solutions of Laplace's equation in both the forms:

$$V = \frac{1}{r^{2n+2}}\left\{\frac{3\cdot 5\cdot 7\cdots 2n+1}{2\cdot 4\cdot 6\cdots 2n} P_{2n+1}\cdot\log r + L_{2n+1}\right\},$$

$$V = r^{2n+1}\left\{\frac{3\cdot 5\cdot 7\cdots 2n+1}{2\cdot 4\cdot 6\cdots 2n} P_{2n+1}\cdot\log r - L_{2n+1}\right\}.$$

L_{2n+1} satisfies the equation:

$$\frac{\partial^2 y}{\partial\theta^2} + \cot\theta\frac{\partial y}{\partial\theta} + (2n+1)(2n+2)y = (4n+3)\frac{3\cdot 5\cdot 7\cdots 2n+1}{2\cdot 4\cdot 6\cdots 2n} P_{2n+1}.$$

This is an equation of the same type as (21), whose solution, subject to the condition of remaining finite on the axis, we have found (23).

16. Another special case of interest is that of a magnetic shell whose strength is proportional to the logarithm of the radius. On the axis the potential is:

$$V = 2\pi kx\int_a^b \frac{\rho\log\rho d\rho}{(x^2+\rho^2)^{3/2}}.$$

Integrating, and developing, we find:

$$\begin{aligned} V = 2\pi k\Big\{ & -\frac{a^2}{x^2}\left(\frac{\log a}{2}-\frac{1}{2^2}\right)+\frac{3a^4}{2x^4}\left(\frac{\log a}{4}-\frac{1}{4^2}\right) \\ & -\frac{3\cdot 5}{2\cdot 4}\frac{a^6}{x^6}\left(\frac{\log a}{6}-\frac{1}{6^2}\right)+\cdots-\frac{x}{b}(\log b+1)+\frac{3x^3}{2b^3}\left(\frac{\log b}{3}+\frac{1}{3^2}\right) \\ & -\frac{3\cdot 5\cdot x^5}{2\cdot 4b^5}\left(\frac{\log b}{5}+\frac{1}{5^2}\right)+\cdots+\log 2x\Big\}. \end{aligned}$$

When equation (31) is applied the result is:

$$\begin{aligned} V = 2\pi k\Big\{ & -\frac{a^2}{r^2}\left(\frac{\log a}{2}-\frac{1}{2^2}\right)P_1+\frac{3a^4}{2r^4}\left(\frac{\log a}{4}-\frac{1}{4^2}\right)P_3 \\ & -\frac{3\cdot 5a^6}{2\cdot 4r^6}\left(\frac{\log a}{6}-\frac{1}{6^2}\right)P_5+\cdots-\frac{r}{b}(\log b+1)P_1 \\ & +\frac{3r^3}{2b^3}\left(\frac{\log b}{3}+\frac{1}{3^2}\right)P_3-\frac{3\cdot 5r^5}{2\cdot 4b^5}\left(\frac{\log b}{5}+\frac{1}{5^2}\right)P_5+\cdots \\ & +\log r\left[\left(1+\frac{1}{2}\right)P_1-\frac{3}{2}\left(\frac{1}{3}+\frac{1}{4}\right)P_3+\frac{3\cdot 5}{2\cdot 4}\left(\frac{1}{5}+\frac{1}{6}\right)P_5-\cdots\right] \\ & +\left(1-\frac{1}{2^2}\right)P_1-\frac{3}{2}\left(\frac{1}{3^2}-\frac{1}{4^2}\right)P_3+\frac{3\cdot 5}{2\cdot 4}\left(\frac{1}{5^2}-\frac{1}{6^2}\right)P_5-\cdots\Big\}. \end{aligned}$$

By (35) we see that the coefficient of $\log r$ is 1. If we write:

$$M_0 = \left(1 - \frac{1}{2^2}\right)P_1 - \frac{3}{2}\left(\frac{1}{3^2} - \frac{1}{4^2}\right)P_3 + \frac{3\cdot 5}{2\cdot 4}\left(\frac{1}{5^2} - \frac{1}{6^2}\right)P_5 - \cdots \tag{42}$$

we see that:

$$\log r + M_0$$

is a solution of Laplace's equation, and we find that M_0 satisfies the equation:

$$\frac{\partial^2 y}{\partial \theta^2} + \cot\theta \frac{\partial y}{\partial \theta} + 1 = 0.$$

The solution of this equation, finite on the axis, is:

$$M_0 = \log(1 + \cos\theta)$$

and it follows that:

$$\log 2 = \left(1 - \frac{1}{2^2}\right) - \frac{3}{2}\left(\frac{1}{3^2} - \frac{1}{4^2}\right) + \frac{3\cdot 5}{2\cdot 4}\left(\frac{1}{5^2} - \frac{1}{6^2}\right) - \cdots, \tag{43}$$

an absolutely convergent series, as is (42).

17. The case just considered is of interest in connection with a distribution of electric currents flowing in concentric circles in a ring-shaped disc such that the current density at any point is inversely proportional to the radius of the point. Such a system of currents is produced in the Corbino effect, when a circular disc carrying a uniform radial current is placed in a normal magnetic field. The magnetic shell equivalent to this distribution is a compound one, consisting of (1) a uniform shell of radius a, strength $k \log b/a$, and (2) a variable ring-shaped shell of radii a and b, and strength $k \log b/\rho$. The potential due to such a compound shell in the region $a < r < b$ is:

$$V = 2\pi k \left\{ \frac{r}{b}P_1 - \frac{1\cdot 1 r^3}{2\cdot 3b^3}P_3 + \frac{1\cdot 1\cdot 3r^5}{2\cdot 4\cdot 5b^5}P_5 - \cdots \right.$$
$$\left. - \frac{1a^2}{2^2r^2}P_1 + \frac{1\cdot 3a^4}{2\cdot 4^2r^4}P_3 - \frac{1\cdot 3\cdot 5a^6}{2\cdot 4\cdot 6^2r^6}P_5 + \cdots - \log\left(\frac{r(1+\cos\theta)}{b}\right)\right\}.$$

If C is the whole circular current, $C = k \log b/a$. When $\theta = \pi/2$, $x = 0$, we have

$$V = 2\pi k \log\frac{b}{\rho},$$

which agrees with (32).

18. We can now get the general results that follow from assuming that the strength of a ring-shaped magnetic shell is equal to

$$\frac{k \log \rho}{\rho^{2n+1}}.$$

If we define:

$$M_{2n} = \sum_{m=0}^{\infty} (-1)^{m+n+1} \frac{3\cdot 5\cdot 7\cdots 2m+1}{2\cdot 4\cdot 6\cdots 2m} \left\{ \frac{1}{(2n-2m-1)^2} - \frac{1}{(2n+2m+2)^2} \right\} P_{2m+1} \tag{44}$$

we find a solution of Laplace's equation:

$$V = \frac{1}{r^{2n+1}} \left\{ \frac{2\cdot 4\cdot 6\cdots 2n}{3\cdot 5\cdot 7\cdots 2n-1} P_{2n} \log r + M_{2n} \right\}.$$

By assuming the strength of the shell to be $k\rho^{2n} \log \rho$, we find a corresponding solution:

$$V = r^{2n} \left\{ \frac{2\cdot 4\cdot 6\cdots 2n}{3\cdot 5\cdot 7\cdots 2n-1} P_{2n} \log r - M_{2n} \right\}.$$

M_{2n} is a solution of the equation:

$$\frac{\partial^2 y}{\partial\theta^2} + \cot\theta \frac{\partial y}{\partial\theta} + 2n(2n+1)y = (4n+1)\frac{2\cdot 4\cdot 6\cdots 2n}{3\cdot 5\cdot 7\cdots 2n-1} P_{2n},$$

an equation which has already been solved. We find further:

$$\begin{aligned} \frac{2\cdot 4\cdot 6\cdots 2n}{1\cdot 3\cdot 5\cdots 2n-1}\log 2 = {} & \frac{2\cdot 4\cdot 6\cdots 2n}{1\cdot 3\cdot 5\cdots 2n-1} \\ & \times \left\{ \frac{1}{2n(2n-1)} + \frac{1}{(2n-2)(2n-3)} + \cdots + \frac{1}{2\cdot 1} \right\} \\ & - \sum_{m=0}^{\infty} (-1)^{m+n+1} \frac{3\cdot 5\cdot 7\cdots 2m+1}{2\cdot 4\cdot 6\cdots 2m} \\ & \times \left\{ \frac{1}{(2n-2m-1)^2} - \frac{1}{(2n+2m+2)^2} \right\}. \end{aligned} \tag{45}$$

The infinite series in this expression is absolutely convergent for all values of n, as is also the infinite series in (44). (43) is the particular case of (45) for $n = 0$. For $n = 1$ we find:

$$2\log 2 = \frac{1}{16} + \frac{3}{2}\left(1 - \frac{1}{6^2}\right) - \frac{3\cdot 5}{2\cdot 4}\left(\frac{1}{3^2} - \frac{1}{8^2}\right) + \frac{3\cdot 5\cdot 7}{2\cdot 4\cdot 6}\left(\frac{1}{5^2} - \frac{1}{10^2}\right) - \cdots.$$

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TOTAL DIFFERENTIABILITY. A CORRECTION.

By E. J. Townsend.

In my paper on Total Differentiability in the September number of the Annals the theorem is incorrect as stated. The boundedness condition should be placed on the second derivative with respect to x or to y and not on the first. However, one may replace this boundedness condition by the more restricted condition of the uniform convergence of either of the difference quotients

$$\frac{f(x_0 + \Delta x, y) - f(x_0, y)}{\Delta x}, \qquad \frac{f(x, y_0 + \Delta y) - f(x, y_0)}{\Delta y}.$$

If we assume the uniform convergence of the first of these difference quotients, the theorem may be stated as follows:

Theorem. *Given a function $f(x, y)$ having at the point (x_0, y_0) the partial derivatives f_x', f_y'; then $f(x, y)$ is totally differentiable at (x_0, y_0) if f_x' is continuous in y at (x_0, y_0) and in the neighborhood of this point the difference quotient*

$$\frac{f(x_0 + \Delta x, y) - f(x_0, y)}{\Delta x} \tag{a}$$

converges uniformly as to Δx.

We have the identity

$$\begin{aligned}\frac{f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0)}{\Delta(x, y)} - \frac{f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0 + \Delta y)}{\Delta(x, y)}& \\ - \frac{f(x_0, y_0 + \Delta y) - f(x_0, y_0)}{\Delta(x, y)} = 0&,\end{aligned} \tag{1}$$

where $\Delta(x, y) \equiv \sqrt{(\Delta x)^2 + (\Delta y)^2}$. Since the partial derivative $f_y'(x_0, y_0)$ exists, we may write

$$\left| \frac{f(x_0, y_0 + \Delta y) - f(x_0, y_0)}{\Delta y} - f_y'(x_0, y_0) \right| < \epsilon,$$

or

$$|f(x_0, y_0 + \Delta y) - f(x_0, y_0) - \Delta y f_y'(x_0, y_0)| < \epsilon |\Delta y|, \qquad |\Delta y| < \delta_1. \tag{2}$$

From the convergence of (a) it follows that f_x' exists for all values of y in the neighborhood of (x_0, y_0) and by hypothesis it is continuous in y at this point. We then have

$$|f_x'(x_0, y_0 + \Delta y) - f_x'(x_0, y_0)| < \frac{\epsilon}{2}, \qquad |\Delta y| < \delta_2. \tag{3}$$

From the uniform convergence of the difference quotient (a) as to Δx for all values of y in the neighborhood of (x_0, y_0), we have

$$\left| \frac{f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0 + \Delta y)}{\Delta x} - f_x'(x_0, y_0 + \Delta y) \right| < \frac{\epsilon}{2}, \qquad |\Delta x| < \lambda, \quad |\Delta y| < \delta_3, \tag{4}$$

where λ is independent of the choice of Δy so long as Δy is less than δ_3 in absolute value. Combining the inequalities (3) and (4) we have

$$\left| \frac{f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0 + \Delta y)}{\Delta x} - f_x'(x_0, y_0) \right| < \epsilon, \qquad |\Delta x| < \lambda, \quad |\Delta y| < \delta_4,$$

where δ_4 is the smaller of the two numbers δ_2, δ_3. We then have

$$|f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0 + \Delta y) - \Delta x f_x'(x_0, y_0)| < \epsilon \cdot |\Delta x|, \qquad |\Delta x| < \lambda, \quad |\Delta y| < \delta_4. \tag{5}$$

Making use of (2) and (5) we have from (1) for $|\Delta x| < \lambda$, $|\Delta y| < \delta$, δ being the smaller of the two numbers δ_1 and δ_4,

$$\left| \frac{f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0) - \Delta x f_x'(x_0, y_0) - \Delta y f_y'(x_0, y_0)}{\Delta(x, y)} \right| < \epsilon \cdot \frac{|\Delta x| + |\Delta y|}{\Delta(x, y)} \leqq \epsilon\sqrt{2}.$$

Hence the first member of this inequality has the limit zero as Δx, Δy approach zero simultaneously, and $f(x, y)$ is totally differentiable at (x_0, y_0) as the theorem states.

A corresponding statement of the theorem may be made involving the continuity of the partial derivative f_y' with respect x at the point (x_0, y_0) and the uniform convergence of the difference quotient

$$\frac{f(x, y_0 + \Delta y) - f(x, y_0)}{\Delta y}$$

in the neighborhood of that point. In the theorem one may replace the continuity of f_x' with respect to y by the continuity of the given function $f(x, y)$ with respect to y for all values of x in the neighborhood (x_0, y_0), since then by virtue of the uniform convergence of the difference quotient (a) the partial derivative f_x' is continuous in y at (x_0, y_0). It follows from this theorem that $f(x, y)$ is totally differentiable at any point (x_0, y_0) of a given region R if f_x', f_y' exist, are continuous in x alone and in y alone and either converges uniformly to its values in the neighborhood of that point.

EXISTENCE THEOREM FOR THE NON-SELF-ADJOINT LINEAR SYSTEM OF THE SECOND ORDER.*

By H. J. Ettlinger.

We shall consider the self-adjoint linear differential equation of the second order,

$$\frac{d}{dx}\left[K(x,\lambda)\frac{du}{dx}\right] - G(x,\lambda)u = 0 \tag{1}$$

together with two linear boundary conditions,

$$U_i = A_{i1}u(a) - A_{i2}K(a)u_x(a) - A_{i3}u(b) + A_{i4}K(b)u_x(b) = 0 \qquad (i = 1, 2) \tag{2}$$

satisfying the conditions:

I. $K(x, \lambda)$ and $G(x, \lambda)$ are continuous, real functions of x in X, $(a \leqq x \leqq b)$, and for all real values of λ in the interval $\Lambda(\Lambda_1 < \lambda < \Lambda_2)$.†

II. $K(x, \lambda)$ is positive everywhere in (X, Λ).

III. The sets of real constants A_{1j} and A_{2j} are not proportional.

IV. For each value of x in X, K and G decrease (or do not increase) as λ increases.

V.‡

$$\lim_{\lambda \to \Lambda_1} - \frac{\min G}{\min K} = -\infty,$$

$$\lim_{\lambda \to \Lambda_2} - \frac{\max G}{\max K} = +\infty.$$

We seek to determine the conditions for the existence of solutions, not vanishing identically, of the system (1), (2). A value of λ, $\lambda = \bar{\lambda}$, for which the system has such a solution is called a *characteristic number*. The problem under investigation is then the following—do there exist characteristic numbers for the system (1), (2)?

We write the array of coefficients of (2) as

$$\begin{pmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \end{pmatrix} \tag{3}$$

and let D_{ij} be the determinant formed by the ith and jth columns. We note first that not all the determinants D_{ij} can vanish, since by condition

* Presented to the Amer. Math. Soc., Dec. 29, 1917.

† In particular Λ may be $-\infty < \lambda < +\infty$.

‡ This condition may be replaced by other sets of conditions. See Bôcher, Leçons sur les Methodes de Sturm, Chap. III, paragraphs 13–15.

III U_1 and U_2 are linearly independent. Moreover, $D_{ij} = -D_{ji}$* and the six determinants D_{ij} satisfy the well-known identity:

(4) $$D_{12}D_{34} + D_{23}D_{14} + D_{42}D_{13} = 0.$$

Let the adjoint boundary conditions† of (2) be

(5) $$V_i = B_{i1}v(a) - B_{i2}K(a)v_x(a) - B_{i3}v(b) + B_{i4}K(b)v_x(b) = 0 \qquad (i = 1, 2).$$

To evaluate V_1 and V_2 we choose U_3 and U_4 defined as in (2), such that U_1, U_2, U_3, and U_4 are linearly independent, or

(6) $$\Delta = \begin{vmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{vmatrix} \neq 0.$$

Green's Theorem gives

(7) $$[K(vu_x - uv_x)]_{x=a}^{x=b} = \sum_{i=1}^{i=4} U_i V_{5-i}.$$

Equating coefficients of corresponding terms in (7), i.e., of $u(a)$, $K(a)u_x(a)$, $u(b)$, $K(b)u_x(b)$ we have

(8) $$\begin{aligned} A_{11}V_4 + A_{21}V_3 + A_{31}V_2 + A_{41}V_1 &= K(a)v_x(a), \\ A_{12}V_4 + A_{22}V_3 + A_{32}V_2 + A_{42}V_1 &= v(a), \\ A_{13}V_4 + A_{23}V_3 + A_{33}V_2 + A_{43}V_1 &= K(b)v_x(b), \\ A_{14}V_4 + A_{24}V_3 + A_{34}V_2 + A_{44}V_1 &= v(b). \end{aligned}$$

Cramer's method yields the solution

(9) $$V_i = \frac{\bar{\Delta}_{5-i}}{\bar{\Delta}} \qquad (i = 1, 2, 3, 4),$$

where $\bar{\Delta}_{5-i}$ is the determinant obtained by replacing the $(5 - i)$th column of $\bar{\Delta}$, the conjugate of Δ, by the right-hand terms of equations (8).

We shall use the following:

Lemma. D_{ij} *is a relative invariant of weight* 1 *of a linear transformation of the coefficients* A_{1j}, A_{2j}.

Proof: Let

$$\begin{aligned} A'_{1k} &= c_1A_{1k} + c_2A_{2k}, \\ A'_{2k} &= d_1A_{1k} + d_2A_{2k}, \end{aligned} \qquad (k = 1, 2, 3, 4)$$

where

* $i = 1, 2, 3, 4$; $j = 1, 2, 3, 4$. In particular $D_{ii} = 0$.

† v is a solution, not vanishing identically, of equation (1). Bôcher, Leçons sur les Methodes de Sturm (1917), p. 28 ff.

$$\delta = \begin{vmatrix} c_1 & c_2 \\ d_1 & d_2 \end{vmatrix} \neq 0.$$

This transformation represents the result of two independent linear combinations of U_1 and U_2. Let D'_{ij} be the determinant formed by the ith and jth columns of the transformed coefficients. Then

$$D'_{ij} = \begin{vmatrix} c_1A_{1i} + c_2A_{2i} & c_1A_{1j} + c_2A_{2j} \\ d_1A_{1i} + d_2A_{2i} & d_1A_{1j} + d_2A_{2j} \end{vmatrix}$$

$$= \begin{vmatrix} c_1 & c_2 \\ d_1 & d_2 \end{vmatrix} \cdot \begin{vmatrix} A_{1i} & A_{1j} \\ A_{2i} & A_{2j} \end{vmatrix} = \delta D_{ij}.$$

It follows immediately from this lemma that if any determinant of (2) vanishes, the corresponding primed determinant will vanish, and if two determinants do not vanish, the sign of their product is an invariant of the transformation. We shall include all boundary conditions of the form (2) under two cases: (1) $D_{12}\cdot D_{34} = 0$, (2) $D_{12}\cdot D_{34} \neq 0$.

Case 1. $D_{12}\cdot D_{34} = 0$.

(*a*) If $D_{12} = 0$, $D_{34} = 0$, then not all the other determinants can vanish. The coefficients of (2) will then be transformed* into

$$(10) \qquad \begin{pmatrix} D_{13} & D_{23} & 0 & 0 \\ 0 & 0 & D_{13} & D_{14} \end{pmatrix}$$

and the boundary conditions (2) are Sturmian.†

(*b*) If $D_{12} \neq 0$, $D_{34} = 0$, we can choose $U_3 = u(b)$ and $U_4 = K(b)u_x(b)$. Then equations (8) yield for the coefficients of (5),

$$(11) \qquad \begin{pmatrix} D_{14} & D_{24} & D_{12} & 0 \\ D_{13} & D_{23} & 0 & D_{21} \end{pmatrix}.$$

(*c*) If $D_{12} = 0$, $D_{34} \neq 0$, we can choose $U_3 = K(a)u_x(a)$ and $U_4 = u(a)$. Then we obtain from (8) for the coefficients of (5),

$$(12) \qquad \begin{pmatrix} D_{43} & 0 & D_{23} & D_{24} \\ 0 & D_{34} & D_{13} & D_{14} \end{pmatrix}.$$

Let $\boldsymbol{D}_{ij}$ be the determinant of the ith and jth columns of the derived adjoint coefficients. It is obvious in *Case* 1. (*a*) that $D_{12}\cdot D_{34} = 0$, since the system is Sturmian. By inspection of (11) and (12) and applying (4), we see that for *Case* 1. (*b*) and (*c*) $\boldsymbol{D}_{12}\cdot\boldsymbol{D}_{34} = 0$. We note, in addition, that the adjoint of $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$ is $\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$. Hence, by applying the lemma, we have in all cases

* $\delta = D_{13} \neq 0$. If $D_{13} = 0$, we may take $D_{24} \neq 0$. If $D_{13} = D_{24} = 0$, then $D_{14} \neq 0$, etc.

† Bôcher, Leçons, p. 60.

THEOREM I. *If for the system* (1), (2) *satisfying conditions I–III,* $D_{12} \cdot D_{34} = 0$, *then for the adjoint system* (1), (5) $\boldsymbol{D}_{12} \cdot \boldsymbol{D}_{34} = 0$; *furthermore if* $D_{12} = 0$, $D_{34} \neq 0$, *then* $\boldsymbol{D}_{12} \neq 0$, $\boldsymbol{D}_{34} = 0$, *and if* $D_{12} \neq 0$, $D_{34} = 0$, *then* $\boldsymbol{D}_{12} = 0$, $\boldsymbol{D}_{34} \neq 0$.

We now consider

Case 2. $D_{12} \cdot D_{34} \neq 0$.

(*a*) If $D_{12} = D_{34}$, the system (1), (2) is self-adjoint, and $D_{12} = \boldsymbol{D}_{12}$, $D_{34} = \boldsymbol{D}_{34}$ or $D_{12} \cdot D_{34} = \boldsymbol{D}_{12} \cdot \boldsymbol{D}_{34}$.

(*b*) If $D_{12} \neq D_{34}$, we shall transform the coefficients of (2) by $\delta = D_{34} \neq 0$ into

$$\begin{pmatrix} D_{14} & D_{24} & D_{34} & 0 \\ D_{13} & D_{23} & 0 & D_{43} \end{pmatrix}. \tag{13}$$

Choose $U_3 = u(b)$ and $U_4 = K(b)u_x(b)$. Then for the coefficients of (5) equations (8) will yield

$$\begin{pmatrix} D_{14} & D_{24} & D_{12} & 0 \\ D_{13} & D_{23} & 0 & D_{21} \end{pmatrix}. \tag{14}$$

By use of (4) we obtain from (14)

$$\boldsymbol{D}_{12} \cdot \boldsymbol{D}_{34} = D_{12}^3 \cdot D_{34}. \tag{15}$$

From (15) we infer immediately that the sign of $D_{12} \cdot D_{34}$ is the same as that of $\boldsymbol{D}_{12} \cdot \boldsymbol{D}_{34}$. Hence, by applying the lemma, we have

THEOREM II. *If for the system* (1), (2) *satisfying conditions I–III,* $D_{12} \cdot D_{34} \neq 0$, *then the sign of* $D_{12} \cdot D_{34}$ *is the same as that of* $\boldsymbol{D}_{12} \cdot \boldsymbol{D}_{34}$ *for the adjoint system* (1), (5).

On the basis of Theorems I and II we classify all systems of the form (1), (2) under the following:

Type I. $D_{12} \cdot D_{34} = 0$;

Type II. $D_{12} \cdot D_{34} > 0$;

Type III. $D_{12} \cdot D_{34} < 0$.

We proceed to state a negative result.

THEOREM III. *The system* (1), (2) *of Type I satisfying conditions I–III, for which*

$$D_{13}^2 + D_{14}^2 + D_{23}^2 + D_{24}^2 = 0,$$

has no characteristic number.

Proof: We see from (11) and (12) that a system (1), (2) of Type I, for which $D_{13} = D_{14} = D_{23} = D_{24} = 0$, can be transformed into either $u(a) = 0$, $u_x(a) = 0$ or $u(b) = 0$, $u_x(b) = 0$. From the properties of the solutions of equation (1) we know that if u and u_x vanish at the same

point, u vanishes identically. Hence the system possesses no characteristic number.

We turn our attention to Type I where $D_{12} \cdot D_{34} = 0$ and $D_{13}^2 + D_{14}^2 + D_{23}^2 + D_{24}^2 \neq 0$. If, as in *Case* 1 (a), $D_{12} = 0$ and $D_{34} = 0$, we have seen from (10) that the system (1), (2) is Sturmian. For this system there exists an infinite set of real characteristic numbers,*

$$\Lambda_1 < \lambda_0 < \lambda_1 < \lambda_2 < \cdots < \Lambda_2.$$

Consider now the *Case* 1 (b), $D_{12} \neq 0$, $D_{34} = 0$, and $D_{24} \neq 0$. Then the coefficients of (2) may be transformed by $\delta = D_{24}$ into

$$\begin{pmatrix} D_{14} & D_{24} & 0 & 0 \\ D_{12} & 0 & D_{32} & D_{42} \end{pmatrix}. \tag{16}$$

It will be convenient to adopt the following notation. Let

$$L[u(x, \lambda)] = \alpha(x)u(x, \lambda) - \beta(x)K(x, \lambda)u_x(x, \lambda), \tag{17}$$

where

$$\alpha(a) = D_{14}, \qquad \alpha(b) = D_{23},$$

$$\beta(a) = D_{24}, \qquad \beta(b) = D_{24}.$$

Then conditions (2) reduce to

$$\begin{aligned} L[u(a, \lambda)] &= 0, \\ L[u(b, \lambda)] &= D_{21}u(a, \lambda). \end{aligned} \tag{18}$$

Let $u_1(x, \lambda)$ be the solution of (1) satisfying

$$u_1(a, \lambda) = D_{24}, \qquad K(a, \lambda)u_{1x}(a, \lambda) = D_{14}.$$

We may assume without loss of generality that D_{14} is positive or else $D_{14} = 0$. The characteristic equation is

$$L[u_1(b, \lambda)] = D_{21} \cdot D_{24}.$$

Let $f(\lambda) = L[u_1(b, \lambda)] + D_{12} \cdot D_{24}$. Now $f(\lambda_0) = D_{12} \cdot D_{24}$ where λ_0 is the first characteristic number of the system consisting of equation (1) together with the boundary conditions,

$$L[u(a, \lambda)] = 0,$$

$$L[u(b, \lambda)] = 0.$$

Moreover

$$\begin{aligned} \lim_{\lambda \to \Lambda_1} f(\lambda) &= \lim_{\lambda \to \Lambda_1} [D_{23}u_1(b, \lambda) - D_{24}K(b, \lambda)u_x(b, \lambda) + D_{12} \cdot D_{24}] \\ &= \lim_{\lambda \to \Lambda_1} u_1(b, \lambda) \left[D_{23} - D_{24}\frac{K(b, \lambda)u_{1x}(b, \lambda)}{u_1(b, \lambda)} + \frac{D_{12} \cdot D_{24}}{u_1(b, \lambda)} \right]. \end{aligned}$$

* Sturm, Jour. de Math., vol. 1, p. 100 ff.

Now

$$\lim_{\lambda \to \Lambda_1} u_1(x, \lambda) = \infty,$$

$$\lim_{\lambda \to \Lambda_1} \frac{K(x, \lambda) u_{1x}(x, \lambda)}{u_1(x, \lambda)} = \infty,$$

for $a < x \leqq b$. Hence for λ sufficiently close to Λ_1, $f(\lambda)$ is negative, or $f(\Lambda_1 + \epsilon) < 0$ if D_{12} be chosen positive and if D_{24} is positive. But $f(\lambda_0)$ is positive. Hence $f(l) = 0$ where $\Lambda_1 < l < \lambda_0$.

If $D_{14} \neq 0$, then we shall transform the coefficients of (2) by $\delta = D_{14}$ into

$$\begin{pmatrix} D_{14} & D_{24} & 0 & 0 \\ 0 & D_{12} & D_{13} & D_{14} \end{pmatrix} \tag{19}$$

or, in terms of the notation (17),

$$L[u(a, \lambda)] = 0,$$

$$L[u(b, \lambda)] = -D_{12} K(a, \lambda) u_x(a, \lambda),$$

where

$$\alpha(a) = D_{14}, \qquad \alpha(b) = D_{13},$$

$$\beta(a) = D_{24}, \qquad \beta(b) = D_{14}.$$

We shall determine $u_1(x, \lambda)$ as the solution of (1) which satisfies

$$u(a, \lambda) = D_{24}, \qquad K(a, \lambda) u_{1x}(a, \lambda) = D_{14},$$

and the characteristic equation becomes

$$L[u_1(b, \lambda)] = -D_{12} \cdot D_{14}.$$

Let

$$f(\lambda) = D_{13} u_1(b, \lambda) - D_{14} K(b, \lambda) u_{1x}(b, \lambda) + D_{12} \cdot D_{14}.$$

Again we have $f(\lambda_0) = D_{12} \cdot D_{14}$, and the sign of $f(\Lambda_1 + \epsilon)$ is that of $-D_{14}$. If $D_{24} \geqq 0$ and D_{12} is chosen positive, if furthermore $D_{14} > 0$, then $f(\lambda_0) > 0$ and $f(\Lambda_1 + \epsilon) < 0$, or $f(l) = 0$ where $\Lambda_1 < l < \lambda_0$.

If $D_{24} = 0$ and $D_{23} \neq 0$, then we shall transform the coefficients of (2) by $\delta = D_{23}$ into

$$\begin{pmatrix} D_{13} & D_{23} & 0 & 0 \\ D_{21} & 0 & D_{23} & 0 \end{pmatrix} \tag{20}$$

or, in terms of the notation (17),

$$L[u(a, \lambda)] = 0,$$

$$L[u(b, \lambda)] = D_{21} u(a, \lambda),$$

where

$$\alpha(a) = D_{13}, \qquad \alpha(b) = D_{23},$$

$$\beta(a) = D_{23}, \qquad \beta(b) = 0.$$

We choose $u_1(x, \lambda)$ the solution of (1) satisfying

$$u_1(a, \lambda) = D_{23}, \qquad K(a, \lambda)u_{1x}(a, \lambda) = D_{13},$$

and the characteristic equation becomes

$$D_{23}u_1(b, \lambda) + D_{12} \cdot D_{23} = 0.$$

Let $f(\lambda) = u_1(b, \lambda) + D_{12}$. If $D_{31} \geqq 0$ and D_{12} is chosen positive and if $D_{32} > 0$, then $f(\lambda_0) > 0$ and $f(\Lambda_1 + \epsilon) < 0$. Hence $f(l) = 0$ where $\lambda_0 < l < \Lambda_1$.

If $D_{14} = 0$ and $D_{13} \neq 0$, then we shall transform the coefficients of (2) by $\delta = D_{13}$ into

$$\begin{pmatrix} D_{13} & D_{23} & 0 & 0 \\ 0 & D_{12} & D_{13} & 0 \end{pmatrix} \tag{21}$$

or, in terms of the notation (17),

$$L[u(a, \lambda)] = 0,$$

$$L[u(b, \lambda)] = D_{21}K(a, \lambda)u_x(a, \lambda),$$

where

$$\alpha(a) = D_{13}, \qquad \alpha(b) = D_{13},$$

$$\beta(a) = D_{23}, \qquad \beta(b) = 0.$$

We determine $u_1(x, \lambda)$ as the solution of (1) satisfying

$$u_1(a, \lambda) = D_{23}, \qquad K(a, \lambda)u_{1x}(a, \lambda) = D_{13},$$

and the characteristic equation becomes

$$D_{13}u_1(b, \lambda) = D_{21} \cdot D_{13}.$$

Let $f(\lambda) = u_1(b, \lambda) + D_{12}$. If $D_{32} \leqq 0$ and if D_{12} is chosen positive and if $D_{31} > 0$, then $f(\lambda_0) > 0$ and $f(\Lambda_1 + \epsilon) < 0$. Hence $f(l) = 0$ where $\lambda_0 < l < \Lambda_1$.

We may dispose at once of *Case* 1 (*c*), $D_{12} = 0$, $D_{34} \neq 0$, by noting that according to Theorem I the various possibilities of *Case* 1 (*c*) are included in the adjoint system of *Case* 1 (*b*). It is a well-known theorem concerning a system (1), (2) and its adjoint (1), (5) that if $\lambda = \bar{\lambda}$ is a characteristic number of the one system, it is also a characteristic number of the other system and the index is the same for both.* In this manner the results for *Case* 1 (*b*) are carried over for *Case* 1 (*c*) with no change.

Consider now *Case* 2, $D_{12} \cdot D_{34} \neq 0$. Let

$$L_i[u(x, \lambda)] = \alpha_i(x)u(x, \lambda) - \beta_i(x)K(x, \lambda)u_x(x, \lambda) \qquad (i = 1, 2)$$

where

* Birkhoff, Trans. Amer. Math. Soc., vol. 9 (1908), p. 373 ff.

$$\alpha_i(a) = A_{i1}, \qquad \beta_i(a) = A_{i2},$$

$$\alpha_i(b) = A_{i3}, \qquad \beta_i(b) = A_{i4}.$$

Then (2) becomes

(22) $$L_i[u(a, \lambda)] = L_i[u(b, \lambda)] \qquad (i = 1, 2).$$

If $D_{12} = D_{34}$, the system is self-adjoint. For such a system (1), (2) satisfying I–V, the writer has proved in an earlier paper* there exists an infinite set of characteristic numbers such that

$$\Lambda_1 < l_0 \leqq l_1 \leqq l_2 \leqq \cdots < \Lambda_2.$$

If $D_{12} \neq D_{34}$, we transform (2) by $\delta = D_{34}$ to

(23) $$\begin{pmatrix} D_{13} & D_{23} & 0 & D_{43} \\ D_{14} & D_{24} & D_{34} & 0 \end{pmatrix}.$$

We take $D_{34} = 1$ and replace conditions (23) by

(24) $$\begin{pmatrix} \sqrt{D_{12}} \cdot \dfrac{D_{31}}{\sqrt{D_{12}}} & \sqrt{D_{12}} \cdot \dfrac{D_{32}}{\sqrt{D_{12}}} & 0 & 1 \\ \sqrt{D_{12}} \cdot \dfrac{D_{14}}{\sqrt{D_{12}}} & \sqrt{D_{12}} \cdot \dfrac{D_{24}}{\sqrt{D_{12}}} & 1 & 0 \end{pmatrix}$$

for Type II and

(25) $$\begin{pmatrix} \sqrt{D_{21}} \cdot \dfrac{D_{31}}{\sqrt{D_{21}}} & \sqrt{D_{21}} \cdot \dfrac{D_{32}}{\sqrt{D_{21}}} & 0 & 1 \\ \sqrt{D_{21}} \cdot \dfrac{D_{14}}{\sqrt{D_{21}}} & \sqrt{D_{21}} \cdot \dfrac{D_{24}}{\sqrt{D_{21}}} & 1 & 0 \end{pmatrix}$$

for Type III.

Let $\nabla = \sqrt{|D_{12}|}$, the positive root, and

$$\bar\alpha_1(a) = \frac{D_{31}}{\nabla}, \qquad \bar\beta_1(a) = \frac{D_{32}}{\nabla}, \qquad \bar\alpha_2(a) = \frac{D_{14}}{\nabla}, \qquad \bar\beta_2(a) = \frac{D_{24}}{\nabla}$$

$$\bar\alpha_1(b) = 0, \qquad \bar\beta_1(b) = 1, \qquad \alpha_2(b) = 1, \qquad \bar\beta_2(b) = 0.$$

Then (24) and (25) may be written

(26) $$\nabla \bar L_i[u(a, \lambda)] = \bar L_i[u(b, \lambda)] \qquad (i = 1, 2)$$

and the adjoint set of conditions are

$$\bar L_i[u(a, \lambda)] = \nabla \bar L_i[u(b, \lambda)] \qquad (i = 1, 2)$$

where for

(27) $$\bar L_i[u(a, \lambda)] = \bar L_i[u(b, \lambda)] \qquad (i = 1, 2)$$

$\bar D_{12} = \bar D_{34} = 1$, and the system (1), (27) is self-adjoint.

* Existence Theorems for the General, Real and Self-Adjoint Linear System of the Second Order. Trans. Amer. Math. Soc., vol. 19 (1918), p. 94.

Let $u_1(x, \lambda)$ and $u_2(x, \lambda)$ be two solutions of (2) satisfying

$$\bar{L}_1[u_1(a, \lambda)] = 0, \qquad \bar{L}_2[u_1(a, \lambda)] = 1,$$

$$\bar{L}_1[u_2(a, \lambda)] = 1, \qquad \bar{L}_2[u_2(a, \lambda)] = 0.$$

Direct computation will show that

$$\bar{L}_1[u_1(b)]\bar{L}_2[u_2(b)] - \bar{L}_1[u_2(b)]\cdot\bar{L}_2[u_1(b)] = -1. \tag{28}$$

Let $u = c_1u_1 + c_2u_2$ be the solution of (1) satisfying system (1), (26). Then the characteristic equation is

$$\psi(\lambda) = \begin{vmatrix} U_1(u_1) & U_2(u_1) \\ U_1(u_2) & U_2(u_2) \end{vmatrix} = 0, \tag{29}$$

or

$$\psi(\lambda) = \begin{vmatrix} -\bar{L}_1[u_1(b, \lambda)] & \nabla - \bar{L}_2[u_1(b, \lambda)] \\ \nabla - \bar{L}_1[u_2(b, \lambda)] & -\bar{L}_2[u_2(b, \lambda)] \end{vmatrix} = 0$$

which simplifies by (28) to

$$\psi(\lambda) = -1 - \nabla^2 - \nabla\{\bar{L}_1[u_2(b, \lambda)] + \bar{L}_2[u_1(b, \lambda)]\} = 0. \tag{30}$$

Now the characteristic equation of the system (1), (27) is

$$\phi(\lambda) = \bar{L}_1[u_2(b, \lambda)] + \bar{L}_1[u_1(b, \lambda)] - 2 = 0.$$

Hence (30) becomes

$$\psi(\lambda) = \nabla\cdot\phi(\lambda) - (1 - \nabla)^2 = 0.$$

Let l_0 be the first characteristic number of (1), (27). Then $\phi(l_0) = 0$ and

$$\psi(l_0) = -(1 - \nabla)^2$$

or $\psi(l_0)$ is negative. Now for λ near Λ_1, the sign of $\psi(\lambda)$ is the same as that of $\phi(\lambda)$. If $D_{24} \neq 0$, the sign of $\phi(\Lambda_1 + \epsilon)$ is that of D_{24}; and if $D_{24} = 0$, the sign is that of D_{14}.* Hence if $D_{24} > 0$ or if $D_{24} = 0$, $D_{14} > 0$, $\psi(\Lambda_1 + \epsilon)$ is positive. Therefore

$$\psi(\bar{l}) = 0$$

where $\Lambda_1 < \bar{l} < l_0$. Since by (4) D_{14} and D_{24} may not both vanish, this exhausts all possibilities. We may now impose on a *non-self-adjoint* system (1), (2) the following condition:

* Cf. Birkhoff, Existence and Oscillation Theorem for a Certain Boundary Value Problem, Trans. Amer. Math. Soc., vol. 15 (1909), p. 268.

VI. *A*. $D_{12} \cdot D_{34} = 0$, and

either (*a*) $D_{24}^2 + D_{14}^2 \neq 0,\ D_{14} \geqq 0,\ D_{24} \geqq 0$,

or (*b*) $D_{24}^2 + D_{14}^2 = 0,\ D_{13}^2 + D_{23}^2 = 0,\ D_{13} \leqq 0,$
$D_{23} \leqq 0$.

B. $D_{12} \cdot D_{34} \neq 0$,

either (*a*) $D_{24} > 0$,

or (*b*) $D_{24} = 0,\ D_{14} > 0$.

We may now state the result in the following form.

EXISTENCE THEOREM. *Every non-self-adjoint system* (1), (2) *satisfying conditions I–V and either VI–A or VI–B has at least one real characteristic number.*

The writer expects to consider in a later paper the system (1), (2) in which A_{ij} are functions of λ and to determine what additional restrictions are necessary to ensure the validity of the Existence Theorem.*

The following six examples illustrate the six types of non-self-adjoint systems for which the Existence Theorem has been established. In addition to the one real characteristic number assured by the Existence Theorem it may be noted that several possibilities arise:

1. The remaining characteristic numbers are real and infinite in number.

2. There is an additional finite number of real characteristic numbers, and the remaining ones are complex and infinite in number.

3. All the remaining characteristic numbers are complex and infinite in number.

4. Every value of λ is a characteristic number.

Example 1. ($D_{12} > 0,\ D_{24} > 0,\ D_{14} = 0$)

$$u'' + \lambda u = 0,$$

$$-\ u'(0) = 0,$$

$$-\ \epsilon u(0) + u'(\pi) = 0, \qquad \epsilon > 0.$$

Let $u(x, \lambda) = \cos \sqrt{\lambda} x$. Then the characteristic equation is

$$\sin \sqrt{\lambda}\pi = -\frac{\epsilon}{\sqrt{\lambda}}.$$

For this case there is always one real characteristic number,

$$\bar{\lambda}_0 = -\frac{v_0^2}{\pi^2} < 0$$

* Since this paper was written, this extension has been presented to the American Mathematical Society, Dec. 31, 1919. See abstract in Bull. Amer. Math. Soc., vol. 26, pp. 267–8.

where v_0 is the root of the equation

$$\sinh v = \frac{\epsilon}{\pi}\frac{1}{v}.$$

There is an infinite number of additional complex characteristic numbers,

$$\bar{\lambda}_k = (2k\pi - iv_0)^2, \qquad \text{where } k = 1, 2, 3, \cdots \text{ and } i = \sqrt{-1}.$$

There will always be an infinite number of real characteristic numbers also, viz., the real roots of

$$\sin \sqrt{\lambda}\pi = -\frac{\epsilon}{\sqrt{\lambda}}.$$

Example 2. $(D_{12} > 0, D_{24} = 0, D_{14} > 0)$.

$$u'' + \lambda u = 0,$$

$$u(0) = 0,$$

$$-\epsilon u'(0) + u'(\pi) = 0, \qquad \epsilon > 0.$$

Let

$$u(x, \lambda) = \frac{\sin \sqrt{\lambda}x}{\sqrt{\lambda}}.$$

Then the characteristic equation is

$$\cos \sqrt{\lambda}\pi = \epsilon.$$

If $\epsilon \leqq 1$,

$$\sqrt{\bar{\lambda}_0} = \frac{1}{\pi}\cos^{-1}\epsilon, \qquad 0 \leqq \cos^{-1}\epsilon < \frac{\pi}{2},$$

$$\bar{\lambda}_0 = \left(\frac{1}{\pi}\cos^{-1}\epsilon\right)^2.$$

where $\bar{\lambda}_0$ is the first characteristic number. The others are

$$\bar{\lambda} = \left(2n \pm \frac{1}{\pi}\cos^{-1}\epsilon\right)^2.$$

If $\epsilon > 1$,

$$\bar{\lambda}_0 = -\left(\frac{\cosh^{-1}\epsilon}{\pi}\right)^2$$

and the remaining characteristic numbers are

$$\bar{\lambda}_k = \left(2k \pm i\frac{\cosh^{-1}\epsilon}{\pi}\right), \qquad \text{where } k = 1, 2, 3, \cdots$$

and $i = \sqrt{-1}$.

Example 3. $(D_{12} > 0,\ D_{14}^2 + D_{24}^2 = 0,\ D_{23} = 0,\ D_{13} < 0)$.

$$u'' + \lambda u = 0,$$

$$u(0) = 0,$$

$$-\epsilon u'(0) + u(\pi) = 0, \qquad \epsilon > 0.$$

Let

$$u(x, \lambda) = \frac{\sin \sqrt{\lambda} x}{\sqrt{\lambda}}$$

and the characteristic equation is

$$\sin \sqrt{\lambda}\pi = \epsilon \sqrt{\lambda}.$$

For $\epsilon \leqq \pi$, $\bar{\lambda}_0 = z_0^2$ where z_0 is the first real solution of the equation

$$\sin \pi z = \epsilon z.$$

For ϵ sufficiently small, there may be other real characteristic numbers.

For $\epsilon > \pi$, $\bar{\lambda}_0 = -\frac{v_0^2}{\pi^2}$ where v_0 is the root of the equation

$$\sinh v = \frac{\epsilon}{\pi} v.$$

For ϵ sufficiently large, other complex roots exist.

Example 4. $(D_{12} > 0,\ D_{14}^2 + D_{24}^2 = 0,\ D_{13} = 0,\ D_{23} < 0)$.

$$u'' + \lambda u = 0,$$

$$\epsilon u(0) - u(\pi) = 0,$$

$$-u'(0) = 0, \qquad \epsilon > 0.$$

Let $u(x, \lambda) = \cos \sqrt{\lambda} x$, and the characteristic equation is

$$\cos \sqrt{\lambda}\pi = \epsilon.$$

The characteristic numbers are those of Example 2.

Example 5. $(D_{12} \cdot D_{34} \neq 0,\ D_{34} = 1,\ D_{24} > 0)$.

$$u'' + \lambda u = 0,$$

$$u(0) + u'(\pi) = 0,$$

$$-\epsilon u'(0) - u(\pi) = 0, \qquad \epsilon > 0.$$

Let

$$u(x, \lambda) = C_1(\lambda)\frac{\sin \sqrt{\lambda} x}{\sqrt{\lambda}} + C_2(\lambda) \cos \sqrt{\lambda} x,$$

and the characteristic equation is

$$\sin \sqrt{\lambda}\pi = -\frac{(1+\epsilon)\sqrt{\lambda}}{1+\epsilon\lambda}.$$

This equation admits real roots.

Example 6. $(D_{12}\cdot D_{34} \neq 0,\ D_{34} = 1,\ D_{24} = 0,\ D_{14} > 0)$.

$$u'' + \lambda u = 0,$$

$$\epsilon u(0) - u(\pi) = 0,$$

$$u'(0) + u'(\pi) = 0, \qquad \epsilon > 0.$$

Let

$$u(x, \lambda) = C_1(\lambda)\frac{\sin \sqrt{\lambda}x}{\sqrt{\lambda}} + C_2(\lambda)\cos \sqrt{\lambda}x,$$

and the characteristic equation is

$$(1 - \epsilon)(1 + \cos \sqrt{\lambda}\pi) = 0.$$

If $\epsilon = 1$, every value of λ is a characteristic number.

If $\epsilon \neq 1$, $\lambda_k = (2k+1)^2$ where $(k = 0, 1, 2, \cdots)$.

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MOTION IN A RESISTING MEDIUM.

By James K. Whittemore.

§ 1. **Introduction.** Most treatises on the dynamics of a particle contain a discussion of the motion of a particle in a straight line under the action of forces expressed in terms of the velocity alone.* It may be remarked at the outset that the same discussion applies to motion in any path provided that the forces considered act along the tangent to the path. The simplest problem in which this condition is realized is motion in a resisting medium, where the force of resistance is opposed to the motion and increases in magnitude with the velocity. This problem is taken as typical in the following discussion. In such motion there is generally a "limiting velocity," sometimes attained, sometimes not attained by the moving particle. I have not found any treatment of the problem where it is shown how the actual velocity approaches the limiting velocity when that limit is not reached, or where the distance covered is compared with the distance which would be covered by a particle moving uniformly with the limiting velocity.† Indeed in so admirable a book as that of Appell, referred to above, there is, in the discussion of the motion of a heavy body falling in a resisting medium, this vague statement, "au bout d'un temps suffisamment longue le mouvement est sensiblement uniforme et de vitesse λ"; λ is the limiting velocity. This gap in the treatment of an important problem of elementary dynamics, I have attempted to fill in this paper. It is proved that if λ is a simple root of $F(v) = 0$, where $F(v)$ is the acceleration of the particle in its path, and when $F(v)$ is subject to certain other conditions generally satisfied in a real problem, $\lambda - v$ approaches zero at least as rapidly as e^{-mt} where t is the time and m a positive constant; that under the same conditions $\lambda t - x$, where x is the distance covered, approaches a limit L, and differs from L by a quantity not greater than a constant multiple of e^{-mt}. When λ is not a simple root of $F(v) = 0$ the results are different.

In § 3 I consider four problems in which there act forces exerted by constant powers, that is forces working at constant rates; such a force is

* See, for example, E. J. Routh, Dynamics of a Particle, Cambridge, 1898, pp. 51, 52, 56–59. P. Appell, Traité de mécanique rationelle, 2d ed., vol. 1, pp. 332–334.

† See however Appell, first edition, pp. 314–318, where these questions are considered for a falling body supposing resistance proportional to velocity.

inversely proportional to the velocity. That the expression for such a force C/v becomes infinite for a vanishing velocity does not invalidate the work, x and v remaining finite for finite t; of course in practice the power is not constant but is applied gradually. I have not found forces of this kind considered in any book on dynamics, yet they enter in real problems. It is certainly more reasonable to suppose that an engine works at a constant rate than to suppose, as is so often done, that it exerts a constant force.

§ 2. **Theory.** Consider a particle P moving with velocity v and tangential acceleration a, and suppose the force along the tangent to its path to depend on v alone, so that $a = F(v)$; let x be the distance covered in the time t, and suppose, for $t = 0$, $v = v_0 \geqq 0$. We assume that $F(v) = 0$ has a positive root λ, and further, if $v_0 < \lambda$, $F(v) = (\lambda - v)^p\varphi(v)$, $p > 0$, $\varphi(v) \geqq m > 0$, for $v_0 \leqq v \leqq \lambda$; $\varphi(\lambda)$ finite; if $v_0 > \lambda$, $F(v) = -(v - \lambda)^p\varphi(v)$, $p > 0$, $\varphi(v) \geqq m > 0$, for $v_0 \geqq v \geqq \lambda$; $\varphi(\lambda)$ finite. Evidently if p is an odd integer we may suppose $F(v)$ to have the same expression for values of v both greater and less than λ.

If $v_0 = \lambda$ we have $a = 0$; the motion is uniform and $x = \lambda t$. We consider in detail the case $v_0 < \lambda$. If $v_0 > \lambda$ we have similar results making the obvious changes in our statements.

Writing $a = dv/dt$, we have

$$(1) \qquad t = \int_{v_0}^{v} \frac{dv}{(\lambda - v)^p\varphi(v)} \leqq \frac{1}{m}\int_{v_0}^{v} \frac{dv}{(\lambda - v)^p}, \qquad v_0 \leqq v \leqq \lambda.$$

If $p < 1$, $v = \lambda$ gives t a finite value; the particle attains the velocity λ in a finite time, thereafter moves uniformly with this velocity. If $p \geqq 1$, $v = \lambda$ in (1) makes t infinite. It follows that as t increases v approaches λ, always increasing, but not reaching that value in a finite time. For this reason λ is called the limiting velocity. If $p > 1$, equation (1) gives

$$t \leqq \frac{(\lambda - v)^{-p+1} - (\lambda - v_0)^{-p+1}}{m(p-1)} < \frac{(\lambda - v)^{-p+1}}{m(p-1)},$$

$$0 < (\lambda - v)^{p-1} < \frac{1}{mt(p-1)}.$$

The case of most interest is that for which λ is a simple root of $F(v) = 0$, $p = 1$. For this case

$$t \leqq \frac{1}{m}\log\frac{\lambda - v_0}{\lambda - v}, \qquad 0 < \lambda - v \leqq (\lambda - v_0)e^{-mt}.$$

Writing now $a = v\,dv/dx$, we have

$$(2) \qquad x = \int_{v_0}^{v} \frac{v\,dv}{(\lambda - v)^p\varphi(v)}.$$

Let P' be a particle moving uniformly with velocity λ in the same path as P, coinciding with P when $t = 0$; then PP', measured along the path, is equal to $\lambda t - x$. From (1) and (2)

$$\lambda t - x = \int_{v_0}^{v} \frac{dv}{(\lambda - v)^{p-1}\varphi(v)}.$$

If $p \geqq 2$, PP' increases indefinitely with t, that is as v approaches λ. If $p < 2$, PP' approaches a finite limit L as t increases, attaining this limit in a finite time only when $p < 1$. We have

$$PP' = \lambda t - x = L - \int_{v}^{\lambda} \frac{dv}{(\lambda - v)^{p-1}\varphi(v)}, \qquad L = \int_{v_0}^{\lambda} \frac{dv}{(\lambda - v)^{p-1}\varphi(v)}.$$

If $p = 1$,

$$0 < L - PP' = \int_{v}^{\lambda} \frac{dv}{\varphi(v)} \leqq \frac{\lambda - v}{m} \leqq \frac{\lambda - v_0}{m} e^{-mt}, \qquad L = \int_{v_0}^{\lambda} \frac{dv}{\varphi(v)}.$$

If $1 < p < 2$,

$$0 < L - PP' \leqq \frac{(\lambda - v)^{2-p}}{m(2 - p)} < \frac{1}{m(2 - p)[mt(p - 1)]^{(2-p)/(p-1)}}.$$

We may state the result of the preceding discussion as follows: for all cases in which $p < 2$ there is a "lost distance" L, lost in attaining full speed λ. In non-mathematical language we may say that after a time "sufficiently long" P is for all subsequent time "practically" at the fixed distance L behind P'.

As previously stated a similar discussion might be given for the case $v_0 > \lambda$: v decreases from v_0 and reaches or approaches the limiting velocity λ; the lost distance L is in this case negative and is really a distance gained, $-L$.

It is of interest to introduce the "time lost" in attaining full speed, $T = L/\lambda$. We do not mean that full speed is attained in the time T, for theoretically the full speed or limiting velocity is never reached unless $p < 1$, but rather we mean that a point of the path "sufficiently" remote from the starting point is reached by P "practically" at an interval T after it is reached by P'. Obviously if $v_0 > \lambda$, T is negative.

§ 3. **Examples.** We consider in this section six examples falling under the preceding discussion.

Example 1. The force is a resistance proportional to any positive power of the velocity.

$$a = -k_1 v^n, \qquad n, k_1 > 0.$$

We have $p = n$, $\lambda = 0$. To have motion we must have $v_0 > 0$. The velocity becomes equal to λ, that is vanishes, in a finite time only if $n < 1$. The particle P' is at rest. There is a finite lost distance, $L =$

$-x$, here negative, only if $n < 2$. This motion is illustrated by a boat drifting in still water with no power applied.

Example 2. There is a constant accelerating force and a resistance proportional to any positive power of the velocity.

$$a = k - k_1 v^n, \qquad n, k, k_1 > 0.$$

Here $p = 1$, and λ is given by $k = k_1 \lambda^n$. The velocity approaches but does not reach the value λ, and there is a finite L. We have

$$F(v) = k_1(\lambda^n - v^n), \qquad \varphi(v) = k_1 \frac{\lambda^n - v^n}{\lambda - v}.$$

Writing $v = \lambda z$, $1 > z \geqq 0$,

$$\varphi(v) = k_1 \lambda^{n-1} \frac{1 - z^n}{1 - z} \geqq k_1 \lambda^{n-1} = m.$$

If, in particular, $v_0 = 0$, we have

$$L = \frac{\rho_n}{k_1 \lambda^{n-2}}, \qquad T = \frac{\rho_n}{k_1 \lambda^{n-1}}, \qquad \rho_n = \int_0^1 \frac{1 - z}{1 - z^n} dz.$$

It may be proved that, as n increases, ρ_n decreases from $\rho_0 = \infty$ to $\rho_\infty = .5$, $\rho_1 = 1$, $\rho_2 = .693$, $\rho_3 = .606$. Further, for $v_0 = 0$,

$$0 < \lambda - v \leqq \lambda e^{-mt} = \lambda e^{-\rho_n t/T} < \lambda e^{-t/2T},$$

$$0 < L - PP' \leqq \frac{\lambda}{m} e^{-mt} = \frac{L}{\rho_n} e^{-\rho_n t/T} < 2L e^{-t/2T}.$$

This case is illustrated by the motion of a body falling in a resisting medium under the action of gravity.

In the following examples we introduce the force exerted by a constant power.

Example 3. There is a constant accelerating power and a constant retarding force.

$$a = \frac{c}{v} - k, \qquad c, k > 0.$$

Here $p = 1$, $c = k\lambda$. The velocity approaches but does not become equal to λ, and there is a finite L. We consider a numerical case: the weight of a train, including locomotive, is 600 tons, the horse power of the locomotive is 500, frictional resistances 15 pounds per ton; the train starts from rest. Taking as units foot, pound, second, $g = 32$,

$$600 \times 2000 \frac{a}{32} = 500 \times \frac{550}{v} - 600 \times 15.$$

We find $\lambda = 275/9$ feet per second, about 21 miles per hour;

$$\varphi(v) = \frac{6}{25v}, \quad m = \frac{6}{25\lambda} = \frac{54}{11 + 25^2}, \quad L = 1945 \text{ feet}, \quad T = 64 \text{ seconds}.$$

$$0 < \lambda - v \leqq \lambda e^{-mt}, \qquad 0 < L - PP' \leqq 3890e^{-mt}.$$

Example 4. There is a constant accelerating power and a resistance proportional to any positive power of the velocity.

$$a = \frac{c}{v} - k_1 v^n, \qquad n, c, k_1 > 0.$$

For this example $p = 1$, and λ is given by $c = k_1\lambda^{n+1}$. The velocity approaches λ without reaching that value and there is a finite L. We have

$$F(v) = k_1\frac{\lambda^{n+1} - v^{n+1}}{v}, \qquad \varphi(v) = k_1\frac{\lambda^{n+1} - v^{n+1}}{v(\lambda - v)}.$$

Setting $v = \lambda z$, $1 > z \geqq 0$,

$$\varphi(v) = k_1\lambda^{n-1}\frac{1 - z^{n+1}}{z(1 - z)} \geqq k_1\lambda^{n-1}\frac{1 - z^{n+1}}{1 - z} \geqq k_1\lambda^{n-1} = m.$$

If in particular $v_0 = 0$ we find

$$L = \frac{\sigma_n}{k_1\lambda^{n-2}}, \qquad T = \frac{\sigma_n}{k_1\lambda^{n-1}}, \qquad \sigma_n = \int_0^1 \frac{z(1 - z)}{1 - z^{n+1}}dz.$$

It may be proved that, for positive n, σ_n is a continuous function of n, decreasing, as n increases, from $\sigma_0 = 1/2$ to $\sigma_\infty = 1/6$, $\sigma_1 = .307$, $\sigma_2 = .247006$, $\sigma_3 = .220$. We find, supposing always $v_0 = 0$,

$$0 < \lambda - v \leqq \lambda e^{-mt} = \lambda e^{-\sigma_n t/T} < \lambda e^{-t/6T},$$

$$0 < L - PP' \leqq \frac{L}{\sigma_n}e^{-\sigma_n t/T} < 6Le^{-t/6T}.$$

We compare the values of L and T in examples 2 and 4, constant force with resistance proportional to v^n, and constant power with resistance proportional to v^n, supposing in both cases $v_0 = 0$:

Example 2, $a = k - k_1v^n$, $k = k_1\lambda^n$,

$$L = \frac{\rho_n}{k_1\lambda^{n-2}} = \frac{\lambda^2}{k}\rho_n = \frac{\rho_n}{k}\left(\frac{k}{k_1}\right)^{2/n}; \qquad \text{for } n = 2,\ L = \frac{.693}{k_1},$$

$$T = \frac{\rho_n}{k_1\lambda^{n-1}} = \frac{\lambda}{k}\rho_n = \frac{\rho_n}{k}\left(\frac{k}{k_1}\right)^{1/n}; \qquad \text{for } n = 1,\ T = \frac{1}{k_1}.$$

Example 4, $a = c/v - k_1v^n$, $c = k_1\lambda^{n+1}$,

$$L = \frac{\sigma_n}{k_1\lambda^{n-2}} = \frac{\lambda^3}{c}\sigma_n = \frac{\sigma_n}{k_1}\left(\frac{k_1}{c}\right)^{(n-2)/(n+1)}; \qquad \text{for } n = 2,\ L = \frac{.247006}{k_1},$$

$$T = \frac{\sigma_n}{k_1\lambda^{n-1}} = \frac{\lambda^2}{c}\sigma_n = \frac{\sigma_n}{k_1}\left(\frac{k_1}{c}\right)^{(n-1)/(n+1)}; \qquad \text{for } n = 1,\ T = \frac{.307}{k_1}.$$

It is remarkable that in both these examples the lost distance L depends only on k_1 and is independent of the constant force or power for $n = 2$ and for no other value of n; that the same is true of the lost time T for $n = 1$.

Example 4 is illustrated by the motion of a steamer in still water driven by an engine working at a constant rate. We note also in this example the formula $c = \lambda^3\sigma_n/L$, from which it follows that, if for a particular ship L is constant for varying power, then the power required to give the ship velocity λ is proportional to λ^3 whatever the value of n.

In the following paragraphs we consider further the motion of a steamer, driven by an engine working at a constant rate, and supposed to start from rest. We speak of λ, the full speed, as if this velocity were actually attained. While that is not true theoretically it is true for practical purposes.

A formula for the indicated horsepower of the engine of a steamer, which is used by marine engineers, is

$$H = \frac{D^{2/3}V^3}{K},$$

where H is the indicated horsepower, D the weight or displacement in long tons, V the velocity in knots, that is in nautical miles of 6,080 feet each per hour, and K a constant depending on the shape and dimensions of the ship. Assuming that the motion comes under example 4 the equation of motion of the ship in foot, pound, second units, $g = 32$, is

$$2240D\frac{a}{32} = 550\frac{H}{v} - Rv^n,$$

where R is the resistance of the water in pounds for unit velocity. Comparing this equation with that of example 4

$$c = \frac{550 \times 32}{2240}\frac{H}{D}, \qquad k_1 = \frac{32}{2240}\frac{R}{D}, \qquad \lambda = \frac{6080}{3600}V.$$

From $c = k_1\lambda^{n+1}$,

$$H = \frac{R}{550}\left(\frac{6080}{3600}V\right)^{n+1} = \frac{D^{2/3}V^3}{K}.$$

It follows that $n = 2$. Since $\sigma_2 = 0.247006$,

$$L = 0.247006 \times \frac{2240}{32 \times 550} \times \left(\frac{6080}{3600}\right)^3 KD^{1/3} = 0.15144KD^{1/3},$$

where L is in feet. It is evident that for a given ship, that is for fixed K, with varying displacement D, L is proportional to $D^{1/3}$. For different

ships of equal displacement L and K are proportional. Since for given D and V the indicated horsepower H varies inversely as K it is evident that the greater K the better is the design of the ship for economy of power. It follows that the greater the distance L lost in acquiring full speed the better is the design. The designs of ships of equal displacement may be compared by measurement of L, or, what amounts to the same thing, K may be computed with known D and L from the last of the preceding formulas. If it is not assumed that $n = 2$, and if the values of c, λ, L are determined σ_n may be found from the equation $c = \lambda^3\sigma_n/L$, then n found from σ_n and the law of resistance determined.

The lost distance L may easily be measured, for L is the difference in the distances run in any period of time τ at full speed and in the period τ from the start, provided that full speed is acquired in the period t from the start. Instead of thus measuring L the time T lost in acquiring full speed may be measured, and L found from $L = \lambda T$. The lost time T is the difference of the times of running ν miles at full speed and of running the first ν miles, provided that full speed is acquired in the first ν miles.

We give finally the numerical value of L computed from

$$L = 0.15144KD^{1/3}$$

for five ships, the values of K and D being taken from C. W. Dyson's Practical Marine Engineering, 7th edition, pp. 616, 617.

1. Yacht,	$D = 366$,	$K = 200$,	$L = 216.7$ feet
2. Small ship,	3,200	230	513.3
3. Ship,	7,243	240	703.2
4. Liner,	15,400	250	942.0
5. Tramp,	8,320	265	813.2.

Example 5. Suppose P a particle of unit mass, moving initially with velocity $v_0 = 2$, acted upon by a unit retarding power, a constant accelerating force 2, and a resistance v. While this is a highly artificial example it is one not impossible to realize physically. It is of a type quite different from the preceding. We have

$$a = -\frac{1}{v} + 2 - v,$$

giving $p = 2$, $\lambda = 1$, $\varphi(v) = 1/v$, $m = 1/2$. From § 2 we have

$$0 < v - 1 < \frac{2}{t},$$

while $P'P$ becomes infinite with t. Carrying out the integration

$$t = \frac{2 - v}{v - 1} - \log (v - 1),$$

from which it may be proved

$$0 < v - 1 \leqq \frac{1 + e^{-2}}{t} < \frac{1.143}{t}.$$

For the distance covered

$$x = \frac{1}{v - 1} - v + 1 - 2 \log (v - 1),$$

and

$$PP' = \lambda t - x = t - x = v - 2 + \log (v - 1).$$

Clearly PP' becomes negatively infinite as t increases indefinitely.

Example 6. The only force is a constant accelerating power, and the particle starts from rest.

We prove this fact: the average velocity measured from the start is always two-thirds of the actual velocity; conversely, if the average velocity of a moving particle measured from rest is always two-thirds of the actual velocity the accelerating force works at a constant rate.

We have $a = c/v$. There is no limiting velocity. Since, for $t = 0$, $x = v = 0$.

$$ct = \frac{v^2}{2}, \qquad cx = \frac{v^3}{3}, \qquad \frac{x}{t} = \frac{2}{3} v.$$

Conversely, if $3x = 2vt$, we have, differentiating with respect to t,

$$3v = 2(v + at),$$

from which

$$\frac{dv}{dt} = a = \frac{v}{2t}, \qquad \frac{v^2}{2} = ct, \qquad \frac{dv}{dt} = \frac{c}{v}.$$

July 24, 1919.

CONTINUOUS MATRICES, ALGEBRAIC CORRESPONDENCES, AND CLOSURE.

By Albert A. Bennett.

I. Introduction.

This article is intended merely to show the intimate relation that exists between three topics not hitherto generally associated. It is first shown how an algebraic correspondence constitutes a type of continuous matrix, and how a continuous matrix serves in turn to suggest at once a weighting in correspondences, a notion that arises in a restricted manner from geometrical studies. The distinction between the inverse and the transposed correspondences, is a feature of the study of correspondences from an algebraic rather than a transcendental viewpoint, being determined not by the character at a point but by the nature of the correspondence as a whole. The condition that an algebraic correspondence has in the sense here used an algebraic inverse, leads at once to finite configurations and problems of closure.

It may be remarked in passing that of the obvious geometric problems of classification of types, one of the most promising, and one as yet wholly untouched, is that of finite variable algebraic configurations, or its equivalent problem of algebraic correspondences with algebraic inverses, the term inverse being understood as here defined. References are contained in the author's paper "Closed Algebraic Correspondences," *Annals of Mathematics*, 2d Series, 18 (1916–17), 200, and need not be repeated here.

II. Weighted algebraic correspondences.

Let A be an irreducible algebraic manifold of one independent complex variable. The projection of A upon the "complex plane" of any complex variable is then a single Riemann surface in the usual sense, while A may be called an irreducible algebraic curve in the domain of birational transformations. Let x denote a variable over A, so that any constant choice assigned to x yields a fixed place on the manifold A. Since the manifold is irreducible, the various possible positions of x can be obtained from a single one by analytic extension. Let B be a second irreducible algebraic manifold with the variable of position, y.

An algebraic correspondence between A and B is a correspondence involving two integers m and n, such that each position x on A determines

n positions y on B, and each position y on B is in turn determined from any one of m positions x on A. A study of correspondences suggests very early the notion of ascribing weight to the various position-pairs in a correspondence. If these all have the same weights the weight might be ascribed to the correspondence as a whole. The weights are frequently positive integers but may have any complex values other than 0. The weight is independent, however, of any continuous variations of the initially determining position so long at least as certain special isolated points be avoided. Thus let x be given at any desired point of A other than the special isolated points explicitly excepted. There are then determined by the correspondence, n points $y_1(x)$, $y_2(x)$, $\cdots$, $y_n(x)$, on B and associated with each pair $(x, y_i(x))$ is a weight c_i independent of variations of x. It may not be possible to assign the weight c_i entirely at pleasure. If by permitting x to make a suitable circuit on A, and avoiding the special points, the x may be brought back to its original place, but two or more y's are interchanged, the interchanged pairs (x, y) will be supposed to have the same weight. The weights are understood indeed to be determined by the x and its associated y's alone, and to be independent of the paths described. It will be assumed that the mere position of x is in all cases sufficient to determine the n positions y. The points x on A that may be associated with a given y on B in the correspondence yield a set of m pairs (x_1, y), (x_2, y), $\cdots$, (x_m, y), each having its own weight. The weight is determined by the pair and the question as to whether the x or the y is regarded as initially assigned is immaterial.

III. Continuous matrices.

A correspondence with weights may be regarded as a continuous real matrix $M(x, y)$. An element or position in the matrix is an arbitrary pair (x, y). The value of an element (x, y) is zero, if the pair are not of positions on A and B respectively, paired under the given correspondence. The value of the element is on the other hand the given weight, if (x, y) form a pair given by the correspondence. A row in the matrix is the totality of pairs (x, y) for which y is fixed, and a column the totality for which x is fixed. Thus a column will contain n, and a row, m non-zero numbers.

Two correspondences between the same two irreducible algebraic manifolds constitute two rectangular matrices $M_1(x, y)$ and $M_2(x, y)$ which may be added or subtracted to yield new matrices or correspondences. A matrix may also be multiplied by a constant. Multiplication by a constant alters only the weights and these proportionately, but does not change the pairs of corresponding elements, providing as usual that the multiplier be not zero.

If a third irreducible algebraic manifold C be taken with z as the corresponding variable of position, a correspondence between B and C will be expressible as a matrix $M(y, z)$. The product $M_1(x, y)$, $M_2(y, z)$ may be formed and a correspondence between A and C will result. In these operations of addition, subtraction, and multiplication the usual rules for finite matrices are observed.

IV. Square matrices.

In particular, correspondences upon a single manifold A will be considered. These may be called square matrices. This will be the case for $M(x, y)$ when B coincides with A. If C be coincident with A then the product $M(x, y)M(y, z)$ will be from x's to z's both on A, while the manifold B may be different. One method of obtaining correspondences upon a single manifold is to start with any $M(x, y)$, where A and B are different, and define $M'(y, x)$ by identity $M'(y, x) \equiv M(x, y)$. If x and y be on the same manifold, that is, if $A \equiv B$, then one may define also $M'(x, y)$, which is then the *transposed* of M. In any case the product $M(x, y)M'(y, x')$ is of the form $M_1(x, x')$ where x and x' are both on A, while $M'(y, x)M(x, y')$ is of the form $M_2(y, y')$ where y and y' are both on B. Both $M_1(x, x')$ and $M_2(y, y')$ are symmetrical, i.e., $M_1 \equiv M_1'$ and $M_2 \equiv M_2'$, and each is obtained from $M(x, y)$ alone. The definition of the product of the square matrices $M_1(x, x')$ and $M_2(x, x')$ is extended so that by M_1M_2 is meant $M_1(x, x'')M_2(x'', x')$.

The most common special matrix is the *identical matrix* for this manifold, of the form $I(x, x')$ where the value of a pair is zero save for $x = x'$, and is then unity.

The notion of a matrix $M(x, y)$ may be extended to some cases in which the correspondence is no longer algebraic, so that in the rows or columns there may be an infinite number of non-vanishing elements. Addition and subtraction present no difficulties but in forming products the additions expressible in the algebraic case in the form

$$\sum_1^p c_{1i}c_{2i}, \qquad i = 1, 2, \cdots, p,$$

are now likely to be of the form of infinite series. The product of two matrices will not be defined unless the series to be summed are (save at most at specified isolated point-pairs with at most an enumerable set of limit pairs) absolutely convergent.

Two square matrices $M_1(x, x')$ and $M_2(x, x')$ are said to be mutually *inverse* if their two products, viz., $M_1(x, x'')M_2(x'', x')$ and $M_2(x, x'')M_1(x'', x')$ are both equal to the identical matrix for this manifold.

It may be noted that so far as the definition is concerned the inverse, M^{-1}, and the transposed M' of a given square matrix M appear to have little in common, M^{-1} being defined by

$$M^{-1}(x, x'')M(x'', x') \equiv M(x, x'')M^{-1}(x'', x') \equiv I(x, x')$$

and M' being defined by $M'(x, x') \equiv M(x', x)$. For algebraic correspondences however these are not unrelated. The algebraic correspondence $M(x, x')$ establishes a relation between x, and x' such that for each x there are corresponding values of x'. What is ordinarily thought of as an inverse is the correspondence determining the values of x that yield a given x', when x' rather than x is given initially. In this sense it is the transposed rather than the inverse matrix that is found. In the terms of functions, if $x' = F(x)$, then $x = F^{-1}(x')$. This implies indeed that in the neighborhood of a point $F(F^{-1}(x)) = F^{-1}(F(x)) = x$. When the total set of values is considered, $F(F^{-1}(x))$ may give other values beside x, as in the case $\log_e(e^x)$ for x complex, giving not only x but also $x + 2n\pi i$ for each integer value of n. Algebraically the resultant of $M(x, x')$ and $M'(x'', x')$ cannot be expected to give merely a power of $(x - x')$ but may contain extraneous factors. An inverse is distinguished from a transposed in that it does not always exist, but if existent accounts for all factors which arise extraneously from the use of the transposed. For algebraic investigation, the study of the inverse naturally presents many interesting features, which cease to be of such importance in the case of infinitely many-valued functions.

For any square matrix M, MM is called the square of M and is written M^2, $MM^2 = M^2M$ is written M^3, etc. If a matrix M has an inverse this is denoted by M^{-1}. A matrix may fail for several reasons to have an inverse, if however an inverse does exist the matrix is said to be *non-singular*.

V. Linear dependence.

A set of k matrices $M_1(x, y)$, $M_2(x, y)$, $\cdots$, $M_k(x, y)$ are defined as being linearly dependent, if there exist k constants, a_1, a_2, $\cdots$, a_k, not all zero such that

$$a_1M_1 + a_2M_2 + \cdots + a_kM_k = 0,$$

where by 0 is meant the matrix all of whose elements have the value zero.

It may happen that a square matrix, $M(x, x')$ singular or not, has the property that the first k powers together with I are linearly dependent so that

$$a_0I + a_1M + a_2M^2 + \cdots + a_kM^k = 0, \qquad a_k \neq 0.$$

The existence of such a relation is not altered if we replace M by any

expression of the form $\lambda M + \mu I$ where $\lambda \neq 0$. The coefficients, a, will of course depend however upon the choice of λ and μ. In the above case we have

$$M(a_k M^{k-1} + \cdots + a_1) = (a_k M^{k-1} + \cdots + a_1)M = -a_0 I$$

so that if a_0 be different from zero,

$$-(a_0 M^{k-1} + \cdots + a_1)/a_0$$

is a matrix inverse to M. Thus if such a linear dependence exists the singular matrices of the pencil, $\lambda M + \mu I$ will correspond to at most a finite number of distinct values of the ratio μ/λ, all others being non-singular. To simplify the study of various problems suggested in this connection we shall introduce the notion of "closure matrices" as distinct from the "correspondence matrices" mentioned heretofore.

VI. Closure.

Let there be a correspondence as considered above from x to x' both on the same irreducible algebraic manifold. Let each x determine n values of x'. The following set S of positions all variable with and determined by x_0 will be considered, x_0; $x_1, x_2, x_3, \cdots, x_n$; $x_{11}, x_{12}, \cdots, x_{1n}, x_{21}, x_{22}, \cdots, x_{2n}, \cdots, x_{n1}, x_{n2}, \cdots, \cdots, x_{nn}$; $x_{111}, x_{112}, \cdots, \cdots, x_{nnn}$; $\cdots$, where $x_{rs\cdots tu}$ is one of the values of x' determined when x is taken as $x_{rs\cdots t}$. A more precise determination of the correspondence from $x_{rs\cdots t}$ to $x_{rs\cdots tu}$ need not be examined, so that the order in which the last set of last subscripts $1, 2, \cdots, n$ is assigned in any case need have no special significance. The totality of *distinct* positions of the above infinite set S will be called the *iterative range* R. For a given x_0 when x is any position of R, the corresponding values of x' are also in R. If no two of the members of S are by the nature of the correspondence identical for all positions of x_0, the set S is said to be *totally open*. If there be pairs that coincide but the reduced set in the iterative range R be yet infinite, the set X is said to be *partially open* and *partially closed*. When R is finite, then S is said to be *totally closed*, or merely *closed*.

The discrete $N(x, x')$ obtained from the correspondence so as to have the same weight as in $M(x, x')$ in point-pairs appearing in both matrices, but where the range of variation of x and x' is now the discrete range R, is called the *closure matrix* of the correspondence. The *closure matrix* will be of finite order and thus of the usual type in algebraic discussion if and only if the correspondence is closed.

VII. Valence.

If K be a closed algebraic correspondence on a single algebraic manifold, then the correspondence $K + vI$ will be singular for certain values of v but non-singular for all other values. Should K have the weight one for all pairs save possibly the pairs (x, x') where $x = x'$, the same will be true for $K + vI$. The choice of v may be regarded as so determined that for $v = 0$, K has the weight zero on all save at most a finite number, c, of "coincident" pairs (x, x'), i.e., where $x = x'$.

Now on a given algebraic manifold A correspondences are possible of certain types but not of others. The study of the manifold on which the correspondence is to be constructed is as necessary as the study of the continuous matrix which is to represent the correspondence when constructed. To be sure, the study of the manifold may be regarded as merely the study of the character of the rows and columns of the continuous matrix, since these are simply duplicates of the manifold.

The matrix being regarded as a two-dimensional manifold D, upon which are traced weighted algebraic curves representing the correspondence, the study of possible correspondences K is the study of possible algebraic curves upon the two-dimensional manifold D. The infinite elements of D will be assumed to be adjoined in such a way as to introduce no singularities when bi-rational transformations are effected either on rows or columns. If p be the genus of A, and c be the number of pairs (x, x') where $x = x'$, then for a correspondence between x, and x' of the form considered, where one x determines n x''s and one x' is determined from m x's, it may be shown that when $p = 0$

$$0 = c - (m + n)$$

but for $p \neq 0$, there may be a number v such that

$$2p \cdot v = c - (m + n).$$

In the simplest cases v is a positive integer, but instances may be readily obtained where v is negative or fractional. This value of v is therefore determined by the correspondence K. In the simpler cases the geometric construction of $K + vI$ with this value of v is particularly simple, and for this reason v as so determined is called the *valence* of K.

The notion of valence arises in geometrical discussions, for the first time, in the following manner. Let C_0 be a fixed curve of genus p, cut by a pencil of curves (C) and defining a correspondence as follows: For any simple point x of C_0 there is a unique curve C of (C) meeting C_0 at x in v coincident points, and in n other variable points x', and further in fixed points only. The correspondence thus established is said to be of valence,

v. If a given point, x', is obtained from n possible choices of x, and if c denote, as above, the number of coincidences, the valence v satisfies the relation $2pv = c - (m + n)$. For an extensive discussion of this relation the reader may refer to H. G. Zeuthen, Lehrbuch der Abzahlenden Geometrie, Chapter IV. Some older writers restrict the term valence to the case of positive integers, all others being termed "singular" correspondences.

VIII. Conclusion.

Necessary and sufficient conditions that a weighted algebraic correspondence has a weighted inverse which is also algebraic, the term "inverse" being used in the complete sense here defined, are the following:

1. The correspondence is completely closed. (This insures that an inverse if existent is not infinitely multiple valued but is algebraic.)

2. The correspondence is non-singular in the sense here used. (This insures the existence of an inverse.)

If the correspondence be singular then other correspondences of the type $K + vI$ where K is as given will be non-singular. However, algebraic considerations suggest a definite value of v, the valence, in any given case, for which the correspondence may or may not be singular.

It should be particularly noted that, while the closure matrix has a determinant, the original matrix will not in general have a determinant in any ordinary sense.

WASHINGTON, D. C.,
November, 1919.

URN SCHEMATA AS A BASIS FOR THE DEVELOPMENT OF CORRELATION THEORY.*

By H. L. Rietz

It is well known that simplicity and precision are gained by the use of urn schemata in establishing various theorems in the theory of probability. The fundamental importance of urn schemata in mathematical statistics is brought out well by Borel† in the statement that "the general problem of mathematical statistics is to determine a system of drawings carried out with urns of fixed composition, in such a way that the results of a series of drawings, lead, with a very high degree of probability, to a table of values identical to the table of observed values."

The expression "urn schemata" is used in the present paper to mean any games of pure chance such as those arranged with balls to be drawn from a bag or urn, or with coins and dice to be thrown.

The urn schema back of the fundamental theorem of Bernoulli, and the urn schema of Poisson's extension of the Bernoulli theorem are so useful in avoiding complicated verbiage that the method of the urn schema is the standard plan of approach to these important theorems.

The theory of correlation that has been much applied in recent years to statistical data has been developed largely as an extension of error theory.‡ It has long seemed to me that it would be important to invent some games of chance that would give a meaning to the correlation coefficient in pure chance, and that would perhaps furnish a basis from which to proceed to develop the theory of correlation. Experiments§ have been performed with dice to show something of the meaning of the correlation coefficient; but the methods were purely empirical and consisted simply in recording the results of a certain number of trials instead of approaching the problem from the standpoint of theoretical probabilities.

It is the main purpose of the present paper to present the results of

* Read before the American Mathematical Society, Sept. 4, 1919.

† Éléments de la Théorie des Probabilités, p. 167; La Hazard, p. 154.

‡ Bravais, Analyse Mathematique sur les Probabilités des Erreurs de Situation d'un Point. Memoirs par divers Savants, 1846.

§ Weldon, Lectures of the Method of Science, Edited by T. B. Strong, Oxford, 1906, pp. 81–100.

Darbishire, Some Tables for Illustrating Statistical Correlation, Memoirs and Proceedings of the Manchester Literary and Philosophical Society, Vol. 51, No. 16, 1907.

devising certain urn schemata which may serve as a starting point of the theory of correlation since a vivid picture is given in this way of the meaning of the coefficient of correlation as related to certain a priori probabilities.

Case I. *Pairs of drawings with balls in common taken at random from the first drawing of a pair. An urn containing white and black balls is so maintained that in drawing a ball the probability of getting a white ball is a constant p and that of getting a black ball is $q = 1 - p$. The first drawing of a pair is to consist of s balls taken one at a time from the urn. The second drawing is to consist of s balls of which t are taken at random from the s first drawn, and $s - t$ are drawn one at a time from the urn. Then the regression is linear, and the coefficient of correlation between the number of white balls in first and second drawings of a pair is t/s, when the frequencies are a set of a priori most probable frequencies.*

To illustrate by means of a simple special case, we exhibit first (Fig. 1) a correlation table for $s = 5$, $t = 3$, $p = \frac{1}{4}$.

Showing Most Probable Frequencies for Pairs of Drawings to Illustrate Case I for $s = 5$, $t = 3$, $p = \frac{1}{4}$

		Number of White Balls in First Drawings of Pairs						Totals
		0	1	2	3	4	5	
Number of White Balls in Second Drawings of Pairs	5				9	6	1	16
	4			81	108	45	6	240
	3		243	648	432	108	9	1,440
	2	243	1,620	1,728	648	81		4,320
	1	1,458	3,159	1,620	243			6,480
	0	2,187	1,458	243				3,888
Totals		3,888	6,480	4,320	1,440	240	16	16,384

Fig. 1.

The table (Fig. 1) exhibits the a priori most probable frequencies when we use as small numbers as possible for frequencies subject to the condition that each frequency is to be an integer. The respective frequencies of 0, 1, 2, 3, 4, 5 white balls in first drawings of pairs are clearly proportional to the terms of the binomial expansion $(\frac{3}{4} + \frac{1}{4})^5$, and such frequencies are shown in the horizontal row of totals at the bottom of the table.

The vertical arrays in the table exhibit frequencies of second drawings of pairs such that the totals of such frequencies satisfy the condition that they are proportional to the terms of the expansion $(\frac{3}{4} + \frac{1}{4})^5$.

From the numbers in the table, we obtain for the correlation coefficient, by the usual method of calculation, the simple result

$$r = 3/5.$$

Proceeding now to the general case, let N_{xy} (Fig. 2) represent the a priori most probable frequency of x white balls in the first drawing and y white balls in the second drawing of a pair. That is, N_{xy} is to be defined for each of the $(s+1)^2$ values obtained by assigning to both x and y the values $0, 1, 2, \cdots, s$. The general method of obtaining these a priori most probable frequencies in convenient form is to first derive in terms of s, t, p, q, x and y the probabilities of obtaining x $(x = 0, 1, 2, \cdots, s)$ white balls followed by y $(y = 0, 1, 2, \cdots, s)$ white balls to make a pair, and then to multiply these probabilities by the smallest positive constant κ that will give products each of which is an integer.

Correlation of Number of White Balls in First and Second Drawings of Pairs of Drawings under Conditions of Case I. Table Showing the a Priori Most Probable Frequencies of κq^{-s} Pairs of Drawings.

		White Balls in First Drawings of Pairs								
		0	1	2	$\cdots$	$x-1$	x	$\cdots$	s	Totals
White Balls in Second Drawings of Pairs	s								$\kappa p^{2s-t}q^{-s}$	$\kappa p^s q^{-s}$
	..	..							..	..
	y	$\kappa_{s-t}C_y q^{s-t-y}p^y$					N_{xy}		..	$\kappa_s C_y p^y q^{-y}$
	$y-1$	..	.. ..						..	..
	..	..	.. ..						..	..
	2	$\kappa_{s-t}C_2 q^{s-t-2}p^2$	.. ..						..	..
	1	$\kappa_{s-t}C_1 q^{s-t-1}p$	.. ..							$\kappa_s C_1 p q^{-1}$
	0	κq^{s-t}	$\kappa(s-t)pq^{s-t-1}$							κ
Totals		κ	$\kappa_s C_1 p q^{-1}$	$\kappa_s C_2 p^2 q^{-2}$	$\cdots$	$\cdots$	$\kappa_s C_x p^x q^{-x}$	$\cdots$	$\kappa p^s q^{-s}$	κq^{-s}

Fig. 2.

In order to express N_{xy} in terms of s, t, x, y, p, and q, we shall explain first the construction of the correlation table outlined in Fig. 2. The first drawings of pairs are simply repeated trials with probability p of success at drawing a white ball and q of failure at doing so. The frequencies may therefore be taken proportional to the terms of the binomial expansion $(q + p)^s$. The frequencies which we find it convenient to use are the terms of this expansion times κq^{-s}.

Corresponding to any number of white balls in first drawings of pairs,

the table (Fig. 2) shows a vertical array for exhibiting the frequencies of various results of second drawings. Thus, corresponding to the drawing of no white balls in first drawings, we have for the a priori most probable frequencies of white balls in second drawings simply κ times the various terms of the expansion $(q + p)^{s-t}$ as shown in the vertical column under the mark 0 for the number of white balls in first drawings.

Consider next the vertical array marked 1. This is to include the a priori most probable values of second drawings that correspond to drawing 1 white ball and $s - 1$ black balls in first drawings. Two cases arise: All of the t balls taken at random from the first drawing may come from the $s - 1$ black balls or $t - 1$ may come from the $s - 1$ black balls and 1 may be the white ball of the first drawing. The number of ways for the first and second of these events to occur is a constant times

$${}_{\overline{s-1}}C_t \quad \text{and} \quad {}_{\overline{s-1}}C_{\overline{t-1}} \quad \text{respectively.}$$

This array consists therefore of two subcolumns of frequencies that may be made up by multiplying the frequencies in the vertical array marked 0 by two numbers proportional to ${}_{\overline{s-1}}C_t$ and ${}_{\overline{s-1}}C_{\overline{t-1}}$ and whose sum is ${}_sC_1 p/q$. That is,

$$\kappa_1({}_{\overline{s-1}}C_t + {}_{\overline{s-1}}C_{\overline{t-1}}) = {}_sC_1 p/q.$$

Since ${}_{\overline{s-1}}C_t + {}_{\overline{s-1}}C_{\overline{t-1}} = {}_sC_t$, we have

$$\kappa_1 = \frac{{}_sC_1}{{}_sC_t}\frac{p}{q}.$$

Hence, the multipliers are

$$\frac{p}{q}\frac{{}_sC_1}{{}_sC_t}\cdot{}_{\overline{s-1}}C_t = \frac{p}{q}(s - t), \qquad \text{and} \qquad \frac{p}{q}\frac{{}_sC_1}{{}_sC_t}\cdot{}_{\overline{s-1}}C_{\overline{t-1}} = \frac{pt}{q}. \tag{A}$$

It should be noted that in viewing the subcolumns from their lower ends upwards, the frequencies different from 0 of the subcolumns begin at 0 white balls for the case in which we use the multiplier $(p/q)(s - t)$ and at 1 white ball for the case in which we use the multiplier $(p/q)t$.

Consider next the vertical array marked 2. It consists of three* subcolumns corresponding to the following ways of drawing t balls from $s - 2$ black balls and 2 white balls: The number of ways in which the t balls can be drawn to include no white ball is ${}_{\overline{s-2}}C_t$, to include one white ball is $2{}_{\overline{s-2}}C_{\overline{t-1}}$, and to include two white balls is ${}_{\overline{s-2}}C_{\overline{t-2}}$. The vertical array marked 2 consists therefore of three subcolumns of frequencies that are made up by multiplying the frequencies in the array marked 0 by

* One of the three columns would vanish if $t < 2$, and a different one if $t > s - 2$. It is to be understood throughout the paper that ${}_mC_n = 0$ if $m < n$.

numbers proportional to ${}_{\overline{s-2}}C_t$, $2\,{}_{\overline{s-2}}C_{\overline{t-1}}$, and ${}_{\overline{s-2}}C_{\overline{t-2}}$ and whose sum is $(p^2/q^2)\,{}_sC_2$. That is,

$$\kappa_2({}_{\overline{s-2}}C_t + 2\,{}_{\overline{s-2}}C_{\overline{t-1}} + {}_{\overline{s-2}}C_{\overline{t-2}}) = {}_sC_2\,\frac{p^2}{q^2}.$$

But since ${}_{\overline{s-2}}C_t + 2\,{}_{\overline{s-2}}C_{\overline{t-1}} + {}_{\overline{s-2}}C_{\overline{t-2}} = {}_sC_t$, we have

$$\kappa_2 = \frac{{}_sC_2}{{}_sC_t}\,\frac{p^2}{q^2}.$$

Hence, the multipliers are

$$\frac{p^2}{q^2}\cdot\frac{{}_sC_2}{{}_sC_t}\cdot{}_{\overline{s-2}}C_t = \frac{p^2}{q^2}\,{}_{\overline{s-t}}C_2, \qquad 2\,\frac{p^2}{q^2}\cdot\frac{{}_sC_2}{{}_sC_t}\cdot{}_{\overline{s-2}}C_{\overline{t-1}} = \frac{p^2}{q^2}\,t(s-t),$$

$$\frac{p^2}{q^2}\,\frac{{}_sC_2}{{}_sC_t}\cdot{}_{\overline{s-2}}C_{\overline{t-2}} = \frac{p^2}{q^2}\,{}_tC_2.$$

Consider similarly the vertical array marked 3. It clearly contains four subcolumns corresponding to 0 white, 1 white, 2 white, and 3 white balls among the t balls in common. Applying the same method as that in obtaining the array marked 2, we find that the multipliers by which to multiply frequencies in the vertical array marked 0 to get the subcolumns of frequencies for the array marked 3 are

$$\frac{p^3}{q^3}\,{}_{\overline{s-t}}C_3, \quad \frac{p^3}{q^3}\,t\,{}_{\overline{s-t}}C_2, \quad \frac{p^3}{q^3}\,{}_tC_2(s-t), \quad \frac{p^3}{q^3}\,{}_tC_3.$$

Similarly, it is easily shown that in the vertical array marked x there are $x+1$ subcolumns of frequencies given by multiplying the frequencies in the vertical array marked 0 by

$$\frac{p^x}{q^x}\,{}_{\overline{s-t}}C_x, \quad \frac{p^x}{q^x}\,t\,{}_{\overline{s-t}}C_{\overline{x-1}}, \quad \frac{p^x}{q^x}\,{}_tC_2\,{}_{\overline{s-t}}C_{\overline{x-2}}, \quad \cdots, \quad \frac{p^x}{q^x}\,{}_tC_{\overline{x-1}}\,{}_{\overline{s-t}}C_1, \quad \frac{p^x}{q^x}\,{}_tC_x, \qquad (B)$$

for the cases of 0 white, 1 white, 2 white, $\cdots$, x white balls respectively among the t balls. Some of the $x+1$ subcolumns may vanish, but this condition is met by the fact that ${}_mC_n = 0$ if $m < n$. Next, form a sum of products of the above-named multipliers by those terms of the expansion $\kappa(q+p)^{s-t}$ that give the frequencies of exactly y white balls in second drawings. This gives the general term

$$\begin{aligned} N_{xy} = \kappa(&{}_{\overline{s-t}}C_x\,{}_{\overline{s-t}}C_y\,q^{s-x-y-t}p^{x+y} + t\,{}_{\overline{s-t}}C_{\overline{x-1}}\,{}_{\overline{s-t}}C_{\overline{y-1}}\,q^{s-x-y-t+1}p^{x+y-1} \\ &+ {}_tC_2\,{}_{\overline{s-t}}C_{\overline{x-2}}\,{}_{\overline{s-t}}C_{\overline{y-2}}\,q^{s-x-y-t+2}p^{x+y-2} + \cdots \qquad (C) \\ &+ {}_tC_x\,{}_{\overline{s-t}}C_{\overline{x-x}}\,{}_{\overline{s-t}}C_{\overline{y-x}}\,q^{s-y-t}p^y). \end{aligned}$$

The sum of frequencies in the horizontal row marked y is given by

$$N_y = N_{0y} + N_{1y} + N_{2y} + \cdots + N_{xy} + \cdots + N_{sy}. \qquad (D)$$

By making $x = 0, 1, 2, 3, \cdots, s$ in the general expression (C) for N_{xy}, we obtain

$$
\begin{aligned}
\frac{1}{\kappa} N_{0y} &= {}_{\overline{s-t}}C_y q^{s-y-t} p^y, \\
\frac{1}{\kappa} N_{1y} &= {}_{\overline{s-t}}C_y q^{s-y-t-1} p^{y+1} {}_{\overline{s-t}}C_1 + t\,{}_{\overline{s-t}}C_{\overline{y-1}} q^{s-y-t} p^y, \\
\frac{1}{\kappa} N_{2y} &= {}_{\overline{s-t}}C_y q^{s-y-t-2} p^{y+2} {}_{\overline{s-t}}C_2 + t\,{}_{\overline{s-t}}C_{\overline{y-1}} q^{s-y-t-1} p^{y+1} {}_{\overline{s-t}}C_1 \\
&\qquad + {}_tC_2\,{}_{\overline{s-t}}C_{\overline{y-2}} q^{s-y-t} p^y, \\
\frac{1}{\kappa} N_{3y} &= {}_{\overline{s-t}}C_y q^{s-y-t-3} p^{y+3} {}_{\overline{s-t}}C_3 + t\,{}_{\overline{s-t}}C_{\overline{y-1}} q^{s-y-t-2} p^{y-2} {}_{\overline{s-t}}C_2 + \cdots, \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\frac{1}{\kappa} N_{\overline{s-t}\,y} &= {}_{\overline{s-t}}C_y\,{}_{\overline{s-t}}C_{s-t} q^{-y} p^{y+s-t} + t\,{}_{\overline{s-t}}C_{\overline{y-1}} q^{-y+1} p^{y+s-t-1} {}_{\overline{s-t}}C_{\overline{s-t-1}} + \cdots \\
\frac{1}{\kappa} N_{s-t+1\,y} &= \qquad\qquad + t\,{}_{\overline{s-t}}C_{\overline{y-1}} q^{-y} p^{y+s-t} {}_{\overline{s-t}}C_{\overline{s-t}} + \cdots, \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots.
\end{aligned}
$$

Adding by columns, and using the fact that

$$q^{s-t} + {}_{\overline{s-t}}C_1 q^{s-t-1} p + {}_{\overline{s-t}}C_2 q^{s-t-2} p^2 + \cdots + p^{s-t} = 1,$$

we have

$$
\begin{aligned}
\frac{1}{\kappa} N_y &= p^y q^{-y} \left({}_{\overline{s-t}}C_y + t\,{}_{\overline{s-t}}C_{\overline{y-1}} + {}_tC_2\,{}_{\overline{s-t}}C_{\overline{y-2}} + \cdots + {}_{\overline{s-t}}C_{\overline{y-t}}\right) \\
&= p^y q^{-y}\,{}_sC_y \text{ by a well-known theorem of combinatorial analysis.}
\end{aligned}
$$

Thus, the sum of frequencies in any horizontal row marked y is

$$N_y = \kappa p^y q^{-y}\,{}_sC_y. \tag{E}$$

Having expressed N_{xy} in terms of s, t, x, y, p, and q, and having determined the nature of the subcolumns that make up vertical arrays, we can express in terms of s and t the correlation coefficient

$$r = \frac{\Sigma(x - \bar{x})(y - \bar{y})}{n\sigma_x\sigma_y}, \tag{F}$$

where x and y are respectively the numbers in first and second drawings of pairs. In this case, $n = \kappa q^{-s}$ as is seen from adding the totals of arrays.

When we multiply the totals of arrays by q^s we obtain the terms of the expansion of $(q + p)^s$ times the multiplier κ.

Hence, it follows that σ_x and σ_y have the same values that they have for a Bernoulli frequency distribution. That is,

$$\sigma_x = \sqrt{spq}, \qquad \sigma_y = \sqrt{spq}. \tag{G}$$

By a somewhat laborious process involving the use of certain theorems of combinatorial analysis, it results that

$$\Sigma(x - \bar{x})(y - \bar{y}) = \kappa tpq^{-s+1},$$

and hence that

$$r = \frac{\Sigma(x - \bar{x})(y - \bar{y})}{n\sigma_x\sigma_y} = \frac{t}{s}. \tag{H}$$

Since I have found a much simpler method of obtaining the value of r than that which involves a separate calculation of $\Sigma(x - \bar{x})(y - \bar{y})$, this simpler process will be presented here. It depends upon the proposition that the means* of the set of vertical arrays (Fig. 2) lie on a straight line of slope t/s.

In order to prove this proposition, we shall show that the difference between the mean M_x of any array marked x and the mean M_{x-1} of an array marked $x - 1$ is t/s.

By considering the subcolumns defined above for an array marked x and making use of the fact that the mean number of successes in a case of $s - t$ trials is $p(s - t)$, when p is the probability of success in a single trial, we can give a formula for the mean M_x derived from the means of the subcolumns weighted with the frequencies in the subcolumns. Thus, for the vertical array marked x, we have

$$\begin{aligned}
{}_sC_x p^x q^{-x} M_x = {}&{}_{s-t}C_x p^x q^{-x} p(s - t) + t\,{}_{s-t}C_{\overline{x-1}}[p(s - t) + 1]p^x q^{-x} \\
&+ {}_tC_2\,{}_{s-t}C_{\overline{x-2}}[p(s - t) + 2]p^x q^{-x} + \cdots \\
&+ {}_tC_{\overline{x-1}}(s - t)[p(s - t) + x - 1]p^x q^{-x} \\
&+ {}_tC_x[p(s - t) + x]p^x q^{-x}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
{}_sC_{\overline{x-1}} p^{x-1} q^{-x+1} M_{x-1} = {}&{}_{s-t}C_{\overline{x-1}}\,p(s - t)p^{x-1}q^{-x+1} \\
&+ t\,{}_{s-t}C_{\overline{x-2}}[p(s - t) + 1]p^{x-1}q^{-x+1} \\
&+ {}_tC_2\,{}_{s-t}C_{\overline{x-3}}[p(s - t) + 2]p^{x-1}q^{-x+1} + \cdots \\
&+ {}_tC_{\overline{x-2}}\,{}_{s-t}C_1[p(s - t) + x - 2]p^{x-1}q^{-x+1} \\
&+ {}_tC_{\overline{x-1}}[p(s - t) + x - 1]p^{x-1}q^{-x+1}.
\end{aligned}$$

* The expression "mean of an array" is used very generally in statistical language as an abbreviation for the "mean of values whose frequencies are exhibited in an array." This is the sense in which "mean of array" is used throughout the present paper.

Then, after some simplification, we obtain

$$M_x - M_{x-1} = \frac{p(s-t)\,[x\,{}_sC_x - (s-x+1)\,{}_sC_{x-1}] - (s-x+1)\,t\,{}_{\overline{s-1}}C_{\overline{x-2}} + tx\,{}_{\overline{s-1}}C_{\overline{x-1}}}{x\,{}_sC_x} = \frac{t}{s}. \quad (I)$$

Thus, when we use the number of white balls in first drawings of pairs as abscissas and the mean values of the number of white balls in the corresponding second drawings of pairs as ordinates, these mean values lie on a straight line (the line of regression) of slope t/s.

But it is well known that*

$$y - \bar{y} = r\frac{\sigma_y}{\sigma_x}(x - \bar{x})$$

is the equation of the line of regression. Since from (G), $\sigma_x = \sigma_y$, we have

$$r = \frac{t}{s} \cdots. \quad (J)$$

This simple result is very interesting for the reason that the correlation coefficient for this urn schema is thus shown to be simply the ratio of the number of balls t in common in the two drawings to the total number s in a drawing.

Case II. *Pairs of drawings with one ball of the most numerous color in first drawings in common.*

We shall consider now an urn schema in which the correlation coefficient does not turn out to be the ratio of the number of balls in common to the total number in a drawing, but in which there is special interest because this case gives us a very simple illustration of non-linear regression from an urn schema, and because r is expressible in simple form in terms of familiar combinations.

An urn containing an equal number of white and black balls is so maintained that in drawing a ball the probability is $\frac{1}{2}$ of getting a white ball. In the first drawing of a pair, s balls are drawn one at a time from the urn giving t of one color and $s - t$ of the other. If $t \neq s - t$, the second drawing of s balls is to consist of $s - 1$ taken one at a time from the urn, and one ball of the color showing the greater number in the first drawing of the pair. If $t = s - t$, the second drawing is to consist like the first of s balls taken one at a time from the urn. Then the regression is non-linear, and the correlation coefficient between the number of white balls in first and second drawings of pairs is $r = {}_sC_{s/2}/2^s$ when s is an even number, and $r = {}_{\overline{s-1}}C_{(s-1)/2}/2^s$ when s is an odd number, under the condition that the frequencies are a set of a priori most probable frequencies.

* See Yule, Introduction to the Theory of Statistics, third edition, p. 171.

In other words, the correlation coefficient is the maximum term of the binomial expansion of $(\frac{1}{2} + \frac{1}{2})^s$ if s is an even number, and the maximum term of the expansion of $(\frac{1}{2} + \frac{1}{2})^{s-1}$ when s is an odd number.

In the table (Fig. 3) is shown for $s = 5$ a set of a priori most probable frequencies with respect to the number of white balls in first and second throws under the conditions specified in Case II. Let coordinates (x, y)

		Number of White Balls in first Drawings of Pairs						
		0	1	2	3	4	5	Totals
Number of White Balls in second drawings of Pairs	5				10	5	1	2^4
	4	1	5	10	40	20	4	2^4 5
	3	4	20	40	60	30	6	2^4 10
	2	6	30	60	40	20	4	2^4 10
	1	4	20	40	10	5	1	2^4 5
	0	1	5	10				2^4
Totals		2^4	2^4 5	2^4 10	2^4 10	2^4 5	2^4	2^9

FIG. 3.

CORRELATION TABLE FOR CASE II WHEN $s = 5$.

represent the number of white balls in first and second drawings of any pair.

The broken line shown in Fig. 3 is the line of regression of second drawings of a pair on first drawings of a pair. Let σ_x and σ_y be the standard deviations of x's and y's respectively. Then it follows at once that

$$\sigma_x = \tfrac{1}{2}\sqrt{s},$$

and

$$\sigma_y = \tfrac{1}{2}\sqrt{s}$$

from the fact that frequencies in vertical arrays are proportional to the coefficients of $1/2^s$ in the terms of the expansion of $(\frac{1}{2} + \frac{1}{2})^s$. The total frequency is $n = 2^{2s-1}$.

Then the problem is to express the correlation coefficient

$$r = \frac{\Sigma\left(x - \frac{s}{2}\right)\left(y - \frac{s}{2}\right)}{n\sigma_x\sigma_y} \tag{K}$$

in as simple a form as possible in terms of s, where the sum Σ is to extend to an entire table of values for s balls in a drawing similar to the table shown in Fig. 3 for $s = 5$.

In a more convenient form, (K) may be written, when s is an odd number, as

$$r = \frac{\displaystyle\sum_{x=0}^{(s-1)/2}\sum_{y=0}^{s-1} {}_sC_x\, {}_{\overline{s-1}}C_y\left(x - \frac{s}{2}\right)\left(y - \frac{s}{2}\right) + \sum_{x=(s+1)/2}^{s}\sum_{y=1}^{s} {}_sC_x\, {}_{\overline{s-1}}C_{y-1}\left(x - \frac{s}{2}\right)\left(y - \frac{s}{2}\right)}{n\sigma_x\sigma_y} \tag{L}$$

$$= \frac{2\displaystyle\sum_{x=0}^{(s-1)/2}\sum_{y=0}^{s-1} {}_sC_x\, {}_{\overline{s-1}}C_y\left(x - \frac{s}{2}\right)\left(y - \frac{s}{2}\right)}{s2^{2s-3}}.$$

Examine the numerator first for the case $x = 0$, $y = 0, 1, 2, \cdots, s - 1$. This subtotal gives the sum

$$\begin{aligned}
&{}_{\overline{s-1}}C_0\left(0 - \frac{s}{2}\right)\left(0 - \frac{s}{2}\right) + \\
&{}_{\overline{s-1}}C_1\left(0 - \frac{s}{2}\right)\left(1 - \frac{s}{2}\right) + \\
&{}_{\overline{s-1}}C_2\left(0 - \frac{s}{2}\right)\left(2 - \frac{s}{2}\right) + \\
&\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\
&\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\
&{}_{\overline{s-1}}C_{\overline{s-1}}\left(0 - \frac{s}{2}\right)\left(s - 1 - \frac{s}{2}\right) \\
\hline
\end{aligned}$$

Adding, we get $\left(0 - \frac{s}{2}\right)[- s2^{s-2} + (s - 1)2^{s-2}] = -\left(0 - \frac{s}{2}\right)2^{s-2}$.

Similarly, add for $x = 1$, and we obtain $- {}_sC_1\left(1 - \frac{s}{2}\right)2^{s-2}$,

for $x = 2$, and we obtain $- {}_sC_2\left(2 - \frac{s}{2}\right)2^{s-2}$,

. . . .

for $x = \frac{s-1}{2}$, and we obtain $- {}_sC_{(s-1)/2}\left(\frac{s-1}{2} - \frac{s}{2}\right)2^{s-2}$.

Collecting for $x = 0, 1, \cdots, (s-1)/2$, we have

$$-\left(s2^{s-2} - \frac{s\,{}_{\overline{s-1}}C_{(s-1)/2}}{2} - s2^{s-2}\right)2^{s-2} = s2^{s-3}{}_{\overline{s-1}}C_{(s-1)/2}.$$

With this value substituted in (L), we obtain

$$r = \frac{{}_{\overline{s-1}}C_{(s-1)/2}}{2^{s-1}}. \tag{M}$$

Similarly, for s an even number

$$r = \frac{{}_sC_{s/2}}{2^s}. \tag{N}$$

If r_s be used for the correlation coefficient under the conditions of these drawings, we note from (M) and (N) that

$$r_{2t} = r_{2t+1}, \text{ where } t \text{ is any positive integer.}$$

The results in Case II differ very much from that in Case I, but it is interesting that the correlation coefficient is simply the maximum term of the binomial expansion $(\frac{1}{2} + \frac{1}{2})^s$ or $(\frac{1}{2} + \frac{1}{2})^{s-1}$ according as s is an even or an odd number. It is also interesting that even this simple Case II does not give linear regression. The means of vertical arrays lie on three straight lines. The means of vertical arrays for $x < s/2$ lie on a horizontal line. The means of vertical arrays for $x > s/2$ lie on another horizontal line one unit from the first line. When s is an odd number, there are two vertical arrays nearest the middle of the table having their means on a straight line of slope 1. When s is an even number, there are three vertical arrays nearest the middle of the table having their means on a straight line of slope $\frac{1}{2}$.

Since this urn schema gives non-linear regression, it may be well to calculate the correlation ratio* η as a substitute for the correlation coefficient. For s an odd number, we obtain

$$\eta = \frac{\sqrt{s}}{s}, \tag{O}$$

and for s an even number

$$\eta = \frac{\sqrt{s}}{s}\sqrt{1 - \frac{{}_sC_{s/2}}{s2^s}}. \tag{P}$$

From a comparison of these results with the corresponding result from Case I, we may note that $r = \eta = 1/s$ if there is one ball in common taken at random from a first drawing as in Case I, whereas when the ball

* Pearson, On the General Theory of Skew Correlation and Non-Linear Regression, Drapers' Company Research Memoirs, 1905, pp. 1–54.

in common is required to be one of the more numerous color as in Case II, we get $\eta = 1/\sqrt{s}$ when s is an odd number and an approximation to this when s is an even number.

Case III. *Consider next a case with a variable number of balls in common. Throw s coins noting the number of heads m and the number of tails $s - m$. If $m > s - m$, leave a number of heads equal to the difference $m - (s - m) = 2m - s$ lying to be counted in the second throw of the pair to be made with the remaining $2s - 2m$ coins. If $s - m > m$, leave the difference $s - 2m$ tails lying to be counted in the next throw with the remaining coins. When $m = s - m$, throw all the s coins for the second throw. Then the regression is linear and the correlation coefficient between the numbers of heads that occur in pairs of throws is*

$$r = \frac{1}{\sqrt{2 - \frac{1}{2^{s-1}}\,{}_{s-1}C_{(s-1)/2}}} \qquad \textit{if } s \textit{ is an odd number,}$$

and

$$r = \frac{1}{\sqrt{2 - \frac{1}{2^{s-1}}\,{}_{s-1}C_{s/2-1}}} \qquad \textit{if } s \textit{ is an even number,}$$

when the frequencies are a set of a priori most probable frequencies.

The following table (Fig. 4) shows for $s = 7$ a set of a priori most probable frequencies with respect to number of heads in first and second throws of pairs of throws of 7 coins in accord with the conditions of Case III.

Similar to the usage in the previous cases, the number of heads in the first throw of a pair will be used as an abscissa and the number in the second throw as an ordinate. Then we have at once

$$\sigma_x = \tfrac{1}{2}\sqrt{s}.$$

The determination of σ_y is considered separately for the cases s even and odd.

I. When s is an odd number. The sum of frequencies in vertical arrays may be represented, beginning at the left, by

$$\kappa,\ \kappa s,\ \kappa_s C_2,\ \kappa_s C_3,\ \kappa_s C_4,\ \cdots,\ \kappa,\ \cdots. \qquad (Q)$$

The plan of finding σ_y^2 is to sum the second moments of all vertical arrays about the mean $s/2$ of the whole vertical distribution. It is simply necessary to weight the squares of standard deviations of binomial distributions with numbers proportional to the numbers (Q). This gives when we weight with numbers $1,\ s,\ {}_sC_2,\ {}_sC_3,\ \cdots,\ 1$,

$$\sigma_y^2 = \frac{2\left[1\left(\frac{s}{2}\right)^2 + s\left\{\frac{2}{4}+\left(\frac{s}{2}-1\right)^2\right\} + {}_sC_2\left\{\frac{4}{4}+\left(\frac{s}{2}-2\right)^2\right\} + {}_sC_3\left\{\frac{6}{4}+\left(\frac{s}{2}-3\right)^2\right\} + \cdots + {}_sC_{(s-1)/2}\left\{\frac{s-1}{4}+\left(\frac{1}{2}\right)^2\right\}\right]}{2^s}$$

$$= \frac{s}{2} - \frac{s}{2^{s+1}}\,{}_{s-1}C_{(s-1)/2},$$

or

$$\sigma_y = \sqrt{\frac{s}{2} - \frac{s}{2^{s+1}}\,{}_{s-1}C_{(s-1)/2}}. \qquad (R)$$

Number of Heads in Second Throws of Pairs	Number of Heads in First Throws of Pairs: 0	1	2	3	4	5	6	7	Totals
7					35	84	112	64	295
6				35	210	336	224		805
5				210	525	504	112		1351
4			84	525	700	336			1645
3			336	700	525	84			1645
2		112	504	525	210				1351
1		224	336	210	35				805
0	64	112	84	35					295
Totals	$2^6\,1$	$2^6\,7$	$2^6\,21$	$2^6\,35$	$2^6\,35$	$2^6\,21$	$2^6\,7$	$2^6\,1$	2^{13}

FIG. 4.

CORRELATION OF NUMBER OF HEADS IN FIRST AND SECOND THROWS OF PAIRS UNDER CONDITIONS OF CASE III FOR $s = 7$.

I calculated the product moments in the numerator of

$$r = \frac{\Sigma\left(x - \frac{s}{2}\right)\left(y - \frac{s}{2}\right)}{n\sigma_x\sigma_y}$$

by obtaining first moments of the frequencies of vertical arrays about the mean of the total distribution and then finding the product moments by multiplying by the appropriate values of $y - s/2$.

But it is simpler to note from the construction of the correlation table as shown, for $s = 7$ in Fig. 4 that the means of arrays lie on a straight line of slope 1 when coordinates are taken as stated above.

Hence,

$$r\frac{\sigma_y}{\sigma_x} = 1,$$

$$r = \frac{\sigma_x}{\sigma_y} = \frac{1}{\sqrt{2 - \frac{1}{2^{s-1}}\,{}_{s-1}C_{(s-1)/2}}}. \qquad (S)$$

It may be of interest to examine this result for a few small odd integers. Thus, when

$$s = 3, \qquad r = \tfrac{1}{3}\sqrt{6} = 0.82,$$

$$s = 5, \qquad r = \tfrac{2}{13}\sqrt{26} = 0.79,$$

$$s = 7, \qquad r = \tfrac{4}{9}\sqrt{3} = 0.77,$$

$$s = 9, \qquad r = \tfrac{8}{221}\sqrt{442} = 0.76,$$

$$\cdots\cdots\cdots\cdots\cdot.$$

II. When s is an even number. By the same general plan as when s is an odd number, we have

$$\sigma_y^2 = \frac{\begin{gathered}1\left(\frac{s}{2}\right)^2 + s\left\{\frac{2}{4} + \left(\frac{s}{2} - 1\right)^2\right\} + \cdots + {}_sC_{(s/2)-1}\left\{\frac{s-2}{4} + \left(\frac{s}{2} - \frac{s}{2} + 1\right)^2\right\} \\ + {}_sC_{s/2}\left\{\frac{s}{4} + \left(\frac{s}{2} - \frac{s}{2}\right)^2 + {}_sC_{(s/2)+1}\left\{\frac{s-2}{4} + \left(\frac{s}{2} - \frac{s}{2} + 1\right)^2\right\}^2 + \cdots\end{gathered}}{2^s}$$

$$= \frac{s}{2} - \frac{s}{2^{s+1}}\,{}_{s-1}C_{(s/2)-1}.$$

Hence,

$$r = \frac{\sigma_x}{\sigma_y} = \frac{1}{\sqrt{2 - \frac{1}{2^{s-1}}\,{}_{s-1}C_{(s/2)-1}}}. \qquad (T)$$

To illustrate for a few special cases, we make $s = 2, 8, 12$.

$$\text{For } s = 2, \qquad r = \frac{\sqrt{6}}{3} = 0.82.$$

$$\text{For } s = 8, \qquad r = \frac{8\sqrt{442}}{221} = 0.76.$$

$$\text{For } s = 12, \qquad r = \frac{32}{1817}\sqrt{1817} = 0.75.$$

It follows at once from (S) and (T) that when $s \to \infty$,

$$r = \frac{\sqrt{2}}{2} = 0.707+.$$

We consider next the following case of throwing two dice with one die taken at random in common.

Case IV. *Two dice are thrown giving a sum x. One of them taken at random is left lying to be counted with the other thrown again. The second trial of a pair thus made gives the sum y. Then the regression is linear and*

	Sums in first Throws of Pairs											
Sums in Second Throws of Pairs	2	3	4	5	6	7	8	9	10	11	12	Totals
12						1	1	1	1	1	1	6
11					1	2	2	2	2	2	1	12
10				1	2	3	3	3	3	2	1	18
9			1	2	3	4	4	4	3	2	1	24
8		1	2	3	4	5	5	4	3	2	1	30
7	1	2	3	4	5	6	5	4	3	2	1	36
6	1	2	3	4	5	5	4	3	2	1		30
5	1	2	3	4	4	4	3	2	1			24
4	1	2	3	3	3	3	2	1				18
3	1	2	2	2	2	2	1					12
2	1	1	1	1	1	1						6
Totals	6	12	18	24	30	36	30	24	18	12	6	216

FIG. 5.

CORRELATION OF SUMS IN FIRST AND SECOND TRIALS OF PAIRS OF TRIALS UNDER CONDITIONS OF CASE IV.

the correlation coefficient between x and y is $\frac{1}{2}$, when the frequencies are a set of a priori most probable frequencies.

The table (Fig. 5) shows a set of a priori most probable frequencies with respect to sums obtained in first and second throws under the conditions of Case IV for having one die in common.

It results at once by a simple calculation that

$$r = \frac{\Sigma(x - \bar{x})(y - \bar{y})}{n\sigma_x\sigma_y} = 0.5.$$

Furthermore, the composition of the table makes it obvious that the regression is linear.

Thus, we find in Case IV an analogy to the result in Case I in that the correlation coefficient is simply the ratio of the one die in common to the two dice thrown. Moreover, the condition is the same in Cases I and IV in that the elements in common are taken at random from first events of pairs.

We consider lastly the following case of pairs of throws of two dice with the die bearing the larger number in the first throw in common.

Case V. *A throw of two dice gives numbers x and y where $x \geqq y$. The die y is left lying to be counted with a second throw of the die x. The second throw of x gives z. Put into correspondence $y + z$ with $x + y$. Then the regression is non-linear and the correlation coefficient of $y + z$ and $x + y$ is*

$$r = \tfrac{3}{181}\sqrt{1086} = 0.5462+$$

and the correlation ratio of $y + z$ on $x + y$ is

$$\eta = 0.5743+$$

when the frequencies are a set of a priori most probable frequencies.

Totals in second Throws with two Dice	Totals in first Throws with two Dice: 2	3	4	5	6	7	8	9	10	11	12	Totals
12						2	2	2	2	2	1	11
11					2	4	4	4	3	2	1	20
10				2	4	6	5	4	3	2	1	27
9			2	4	5	6	5	4	3	2	1	32
8		2	3	4	5	6	5	4	3	2	1	35
7	1	2	3	4	5	6	5	4	3	2	1	36
6	1	2	3	4	5	4	3	2	1			25
5	1	2	3	4	3	2	1					16
4	1	2	3	2	1							9
3	1	2	1									4
2	1											1
Totals	6	12	18	24	30	36	30	24	18	12	6	216

Fig. 6.

Correlation of Totals Thrown in First and Second Throws with Two Dice—the Larger Number Thrown in a First Throw being Counted in Common. Case V.

The table (Fig. 6) shows a set of a priori most probable frequencies of totals in first and second throws of pairs of throws with two dice under the conditions stated in Case V. It is obvious from the location of means of vertical arrays in the table that the regression is far from linear. The coördinates of the means of arrays are

$$(2, 4.5), (3, 5.5), (4, 6\tfrac{1}{6}), (5, 7), (6, 7.7), (7, 8.5), (8, 8.7), (9, 9), (10, 9\tfrac{1}{6}),$$
$$(11, 9.5), (12, 9.5).$$

Since the regression is non-linear, we have calculated the correlation ratio as well as the correlation coefficient, the former being, in general, the more appropriate function for the characterization of correlation in a case of non-linear regression given by a single-valued function. Case V is of special interest because of its bearing on the view that all very simple cases of correlation lead to linear regression. We have in this simple case of pure chance a very significant departure from linear regression.

In conclusion, the results of this paper make clear the meaning of the correlation coefficient for certain urn schemata, and indicate that the elements of the theory of correlation may be developed from such urn schemata as we have devised. Such a development from a priori probabilities seems decidedly less empirical than existing developments. It may be urged against a development from a priori probabilities that it neglects fluctuations in random sampling. The answer to this criticism is that we may actually carry out the corresponding experiments with the urn schemata when we wish to include fluctuations in sampling.

THE UNIVERSITY OF IOWA.

ON PSEUDO-RESOLVENTS OF LINEAR INTEGRAL EQUATIONS IN GENERAL ANALYSIS.

By T. H. Hildebrandt.

The purpose of this note is to extend the theory of pseudo-resolvents in linear integral equations as given by W. A. Hurwitz* to the General Analysis situation. This extension is a perfectly obvious one, if the basis is that used by Moore† for the Hilbert-Schmidt theory. For in this theory it is possible to carry through the argument exactly as given by Hurwitz. However, the Hurwitz theory is intended to replace the theory based on Fredholm minors, and so it would be desirable to operate on the basis which Moore uses for the Fredholm theory. We show that this is possible if we add an additional postulate on the J-operation in the case of one theorem: Theorem 5.

We assume that we are working in a basis (Σ_6):‡

$$\left(\mathfrak{A}; \begin{matrix} \bar{\mathfrak{P}},\ \bar{\mathfrak{M}},\ \mathfrak{K} = (\bar{\mathfrak{M}}\hat{\mathfrak{M}})_*, \\ \hat{\mathfrak{P}},\ \hat{\mathfrak{M}},\ \check{\mathfrak{K}} = (\hat{\mathfrak{M}}\bar{\mathfrak{M}})_*, \end{matrix} \qquad J \text{ on } \check{\mathfrak{K}} \text{ to } \mathfrak{A}\right)$$

where $\mathfrak{A}$ is the class of complex numbers, $\bar{\mathfrak{P}}$ and $\hat{\mathfrak{P}}$ are general classes of elements: $\bar{p}$ and $\hat{p}$; $\bar{\mathfrak{M}}$ and $\hat{\mathfrak{M}}$ are classes of functions $(\bar{\mu}, \bar{\xi}, \bar{\eta};\ \hat{\mu}, \hat{\xi}, \hat{\eta})$ on $\bar{\mathfrak{P}}$, $\hat{\mathfrak{P}}$ respectively to $\mathfrak{A}$, and have the properties L, C and D; the operator J has the properties L and M, where in accordance with a recent simplification of Professor Moore's, *J has the property M*, if for every ν the values of $|J\nu_1|$ for the functions $|\nu_1| \leqq |\nu|$ have an upper bound. If $M\nu$ is defined to be the least upper bound of such values then M is a functional operation on $\mathfrak{N}$ to the non-negative real part of $\mathfrak{A}$, of such a nature that

$$\text{if } |\nu_1| \leqq |\nu| \text{ then } |J\nu_1| \leqq M\nu \text{ and } M\nu_1 \leqq M\nu.$$

We shall assume the following results derivable from the existence of the Fredholm determinant and first minor in this basis:

* Cf. Transactions of the American Mathematical Society, vol. 13 (1912), pp. 405–418; 16 (1915), pp. 121–133. The desirability of such an extension has recently been mentioned by Evans: Cambridge Colloquium, p. 115. The present note gives a general case including the two papers of Hurwitz as special cases.

† Cf. Bulletin of the American Mathematical Society, ser. 2, vol. 18 (1912), p. 361, and Proceedings of Fifth International Congress, Cambridge, vol. I, p. 232.

‡ Cf. Moore, loc. cit., p. 352.

(A) If the equation $\bar{\xi} + J\kappa\bar{\xi} = 0$ has a solution $\bar{\xi}$ which is not identically zero, then the adjoint equation $\hat{\xi} + J\hat{\xi}\kappa = 0$ has a solution $\hat{\xi}$ not identically zero, and conversely.

(B) The Fredholm determinant $F_0(\kappa; z)$ of $z\kappa$ is an entire function of z. If κ_1 and κ_2 are such that $J\kappa_1\kappa_2 = 0$, then

$$F_0(\kappa_1 + \kappa_2; z) = F_0(\kappa_1; z)F_0(\kappa_2; z).$$

We also note the following well-known result from the theory of linear independence of functions:

(C) A necessary and sufficient condition that n functions $\mu_1, \cdots, \mu_n$ on $\mathfrak{P}$ to $\mathfrak{A}$ be linearly independent, is that there exist values $p_1, \cdots, p_n$ of the range $\mathfrak{P}$ such that the determinant of the elements $\mu_i(p_j)$ is not zero. As a consequence, if $\mu_1, \cdots, \mu_n$ are linearly independent, it is possible to find linear combinations μ_{0i} of the μ_i such that $\mu_{0i}(p_j) = \delta_{ij}$ (where δ_{ij} is the Kronecker δ, i.e., zero for $i \neq j$ and unity for $i = j$) and such that the set μ_{0i} is linearly equivalent to the set μ_i.

We then have the following theorems:

1. *The number of linearly independent solutions of* $\bar{\xi} + J\kappa\bar{\xi} = 0$ *is finite.*

Let $\bar{\mu}_1, \cdots, \bar{\mu}_n$ be a set of linearly independent solutions of $\bar{\xi} + J\kappa\bar{\xi} = 0$, which according to ($C$) can be chosen in such a way that there exist $\bar{p}_j$ for which $\bar{\mu}_i(\bar{p}_j) = \delta_{ij}$. Then if $\kappa(\bar{p}_j) = \hat{\mu}_j$ (which is obviously a function of the class $\hat{\mathfrak{M}}$), we have:

$$J\hat{\mu}_i\bar{\mu}_j = J\kappa(\bar{p}_i)\bar{\mu}_j = -\bar{\mu}_i(\bar{p}_j) = -\delta_{ij}.$$

Let

$$\kappa_1 = \kappa - \Sigma_1{}^n\bar{\mu}_i\hat{\mu}_i \qquad \text{and} \qquad \kappa_2 = \Sigma_1{}^n\bar{\mu}_i\hat{\mu}_i.$$

Then

$$J\kappa_1\kappa_2 = J(\kappa - \Sigma\bar{\mu}_i\hat{\mu}_i)\Sigma\bar{\mu}_j\hat{\mu}_j = 0.$$

Hence

$$\begin{aligned} F_0(\kappa; z) &= F_0(\kappa - \Sigma\bar{\mu}_i\hat{\mu}_i + \Sigma\bar{\mu}_i\hat{\mu}_i; z) \\ &= F_0(\Sigma\bar{\mu}_i\hat{\mu}_i; z)F_0(\kappa - \Sigma\bar{\mu}_i\hat{\mu}_i; z) = (1 - z)^nF_0(\kappa - \Sigma\bar{\mu}_i\hat{\mu}_i; z). \end{aligned}$$

Since $F_0(\kappa; z)$ is an entire function of z, it can have the root $z = 1$ at most a finite number of times. Hence n is bounded.

A similar result holds for the equation $\hat{\xi} + J\hat{\xi}\kappa = 0$.

2. *The equation* $\bar{\xi} + J\kappa\bar{\xi} = 0$ *and its adjoint* $\hat{\xi} + J\hat{\xi}\kappa = 0$ *have the same number of linearly independent solutions.*

Let $\bar{\mu}_{0i}$ $(i = 1, \cdots, n)$ be a complete set of linearly independent solutions of $\bar{\xi} + J\kappa\bar{\xi} = 0$ of such a nature that $\bar{\mu}_{0i}(\bar{p}_j) = -\delta_{ij}$, and let $\kappa(\bar{p}_j) = \hat{\mu}_j$, so that $J\hat{\mu}_i\bar{\mu}_{0j} = \delta_{ij}$. Similarly let $\hat{\mu}_{0i}$ $(i = 1, \cdots, m)$ be a complete set of linearly independent solutions of $\hat{\xi} + J\hat{\xi}\kappa = 0$, such that

$\hat{\mu}_{0i}(\bar{p}_j) = -\delta_{ij}$ and let $\kappa(\hat{p}_j) = \bar{\mu}_j$ so that $J\hat{\mu}_{0i}\bar{\mu}_j = \delta_{ij}$. Suppose $m < n$. Then set $\kappa_0 = \kappa + \Sigma_1{}^m\bar{\mu}_i\hat{\mu}_i$. Then by following out the method used by Hurwitz, it can be shown that the equation $\hat{\xi} + J\hat{\xi}\kappa_0 = 0$ has no non-identically zero solution, while $\bar{\xi} + J\kappa_0\bar{\xi} = 0$ has at least the solution $\bar{\mu}_{0m+1}$, which by the result (A) is impossible.

As an immediate consequence it follows that κ_0 has a reciprocal λ. If we replace κ_0 by its equivalent in the reciprocal relations which it satisfies, and make a slight transformation, we find that

3. $\kappa + \lambda + J\kappa\lambda + \Sigma_1{}^n\bar{\mu}_i\hat{\mu}_{0i} = 0,$

$\kappa + \lambda + J\lambda\kappa + \Sigma_1{}^n\bar{\mu}_{0i}\hat{\mu}_i = 0.$

By using these relations, we prove at once:

4. *A necessary and sufficient condition that* $\bar{\xi} + J\kappa\bar{\xi} = \bar{\eta}$ *shall have a solution* $\bar{\xi}$ *is that* $J\hat{\mu}_{0i}\bar{\eta} = 0$ *for every* i. *The solution is expressible in the form*

$$\bar{\xi} = \bar{\eta} + J\lambda\bar{\eta} + \Sigma c_i\bar{\mu}_{0i},$$

where the c_i *are arbitrary constants.*

Hurwitz's definition of pseudo-resolvents in this situation reads:

λ *is a pseudo-resolvent of* κ if the solution of the equation

$$\bar{\xi} + J\kappa\bar{\xi} = \bar{\eta}$$

in which $J\hat{\mu}_{0i}\bar{\eta} = 0$ for $i = 1, \cdots, n$ ($\hat{\mu}_{0i}$ being any complete set of linearly independent solutions of the adjoint homogeneous equation $\hat{\xi} + J\hat{\xi}\kappa = 0$) is expressible in the form

$$\bar{\xi} = \bar{\eta} + J\lambda\bar{\eta} + \Sigma c_i\bar{\mu}_{0i}$$

where $\bar{\mu}_{0i}$ constitute a complete set of solutions of the homogeneous equation $\bar{\xi} + J\kappa\bar{\xi} = 0$.

For the following theorem it is desirable to introduce another postulate on the J-operation,* which is a generalization of the definite property of the integral. It is as follows:

If $J\hat{\mu}_0\bar{\mu} = 0$ for every $\bar{\mu}$ of the class $\bar{\mathfrak{M}}$, then $\hat{\mu}_0 = 0$.

This property is equivalent to:

If $J\kappa_0\bar{\mu} = 0$ for every $\bar{\mu}$ of the class $\bar{\mathfrak{M}}$, then $\kappa_0 = 0$. If J has this property, we shall say that it is *definite on the left.* Similarly J is *definite on the right*, if $J\hat{\mu}\bar{\mu}_0 = 0$ or $J\hat{\mu}\kappa_0 = 0$ for every $\hat{\mu}$ of the class $\hat{\mathfrak{M}}$ has as consequence $\bar{\mu}_0 = 0$ and $\kappa_0 = 0$, respectively. If J is definite on the right and left, then it is said to be *definite.*

We then have the theorem:

5. *If J is definite on the left, a necessary and sufficient condition that* λ

* Cf. Transactions of the American Mathematical Society, vol. 19 (1918), p. 101.

be a pseudo-resolvent of κ is that

$$\kappa + \lambda + J\kappa\lambda + \Sigma_1^n \bar{\mu}_i \hat{\mu}_{0i} = 0,$$

$$\kappa + \lambda + J\lambda\kappa + \Sigma_1^n \bar{\mu}_{0i} \hat{\mu}_i = 0,$$

where the $\bar{\mu}_{0i}$ and $\hat{\mu}_{0i}$ constitute complete sets of linearly independent solutions of the homogeneous equations $\bar{\xi} + J\kappa\bar{\xi} = 0$ and $\hat{\xi} + J\hat{\xi}\kappa = 0$, respectively, and $J\hat{\mu}_i\bar{\mu}_{0j} = J\hat{\mu}_{0i}\bar{\mu}_j = \delta_{ij}$.

The sufficiency of this condition without the additional postulate on the J is shown as in Theorem 4.

In order to show the necessity, we make a study of the function:

$$\kappa_1 = \kappa + \lambda + J\kappa\lambda.$$

It has the following properties:

(*a*) $\hat{\mu}_{0i} + J\hat{\mu}_{0i}\kappa_1 = 0$ for every i.

For

$$J\hat{\mu}_{0i}\kappa_1 = J\hat{\mu}_{0i}\kappa + J\hat{\mu}_{0i}\lambda + JJ\hat{\mu}_{0i}\kappa\lambda = J\hat{\mu}_{0i}\kappa = -\hat{\mu}_{0i}.$$

As a consequence the equation $\hat{\xi} + J\hat{\xi}\kappa_1 = 0$ has at least the n linearly independent solutions $\hat{\mu}_{01}, \cdots, \hat{\mu}_{0n}$.

(*b*) $J\hat{\mu}_{0i}(\kappa - \kappa_1) = 0$ for every i.

This is a direct consequence of (*a*).

(*c*) If $\bar{\mu}$ is such that $J\hat{\mu}_{0i}\bar{\mu} = 0$ for every i, then $J\kappa_1\bar{\mu} = 0$ and conversely. For suppose $J\hat{\mu}_{0i}\bar{\mu} = 0$ for every i. Then the equation $\bar{\mu} = \bar{\xi} + J\kappa\bar{\xi}$ has a solution:

$$\bar{\xi} = \bar{\mu} + J\lambda\bar{\mu} + \Sigma c_i\bar{\mu}_{0i}.$$

If we substitute this value of ξ in the equation $\bar{\mu} = \bar{\xi} + J\kappa\bar{\xi}$, we find that $J\kappa_1\bar{\mu} = 0$. The converse is obtained from (*a*) by multiplying by $\bar{\mu}$ and operating with J.

(*d*) $J\kappa_1(\kappa - \kappa_1) = 0$.

This is an immediate consequence of (*b*) and (*c*).

(*e*) The function $J\kappa_1\kappa_1$ satisfies the condition:

$$J\kappa_1\kappa_1 + J(J\kappa_1\kappa_1)\kappa = 0.$$

For consider the expression $\kappa_1 + J\kappa_1\kappa$. We have:

$$J\hat{\mu}_{0i}(\kappa_1 + J\kappa_1\kappa) = J\hat{\mu}_{0i}\kappa_1 + JJ\hat{\mu}_{0i}\kappa_1\kappa = -\hat{\mu}_{0i} + \hat{\mu}_{0i} = 0.$$

Applying the result (*c*) we find:

$$J\kappa_1(\kappa_1 + J\kappa_1\kappa) = 0 \qquad \text{or} \qquad J\kappa_1\kappa_1 + J(J\kappa_1\kappa_1)\kappa = 0.$$

As a consequence

(*f*) $J\kappa_1\kappa_1 = \Sigma_1{}^n\bar{\mu}_i\hat{\mu}_{0i}$, and as a matter of fact $J\hat{\mu}_{0i}\bar{\mu}_j = \delta_{ij}$. For

$$JJ\hat{\mu}_{0j}\kappa_1\kappa_1 = \Sigma J\hat{\mu}_{0j}\bar{\mu}_i\hat{\mu}_{0i}.$$

But $JJ\hat{\mu}_{0j}\kappa_1\kappa_1 = \hat{\mu}_{0j}$, and the $\hat{\mu}_{0j}$ are linearly independent. Hence

$$J\bar{\mu}_{0i}\bar{\mu}_j = \delta_{ij}.$$

(*g*) The functions $\bar{\mu}_i$ satisfy the homogeneous equation $\bar{\xi} + J\kappa_1\bar{\xi} = 0$. For

$$JJ\kappa_1\kappa_1\kappa_1 = \Sigma J\kappa_1\bar{\mu}_i\hat{\mu}_{0i} = \Sigma\bar{\mu}_iJ\hat{\mu}_{0i}\kappa_1 = -\Sigma\bar{\mu}_i\hat{\mu}_{0i},$$

according as we multiply the expression $J\kappa_1\kappa_1$ by κ_1 on the left or right and operate with J. If we multiply the equality:

$$\Sigma J\kappa_1\bar{\mu}_i\hat{\mu}_{0i} = -\Sigma\bar{\mu}_i\hat{\mu}_{0i}$$

by $\bar{\mu}_j$ on the right and operate with J, we obtain:

$$J\kappa_1\bar{\mu}_j = -\bar{\mu}_j.$$

(*h*) Every solution of $\bar{\xi} + J\kappa_1\bar{\xi} = 0$ is a linear combination of the $\bar{\mu}_i$ and every solution of $\hat{\xi} + J\hat{\xi}\kappa_1 = 0$ is a linear combination of the $\hat{\mu}_{0i}$.

For every solution of $\bar{\xi} + J\kappa_1\bar{\xi} = 0$ is also a solution of

$$\bar{\xi} - J(J\kappa_1\kappa_1)\bar{\xi} = 0,$$

and every solution of $\hat{\xi} + J\hat{\xi}\kappa_1 = 0$ is also a solution of $\hat{\xi} - J\hat{\xi}(J\kappa_1\kappa_1) = 0$. If we replace $J\kappa_1\kappa_1$ by its value $\Sigma_1{}^n\bar{\mu}_i\hat{\mu}_{0i}$, the result is immediate. It follows from this result that the $\bar{\mu}_i$ and $\hat{\mu}_{0i}$ constitute complete sets of solutions of the homogeneous equations in κ_1.

(*i*) $J(\kappa_1 + \Sigma\bar{\mu}_i\hat{\mu}_{0i})\bar{\mu} = 0$ for every $\bar{\mu}$ of the class $\bar{\mathfrak{M}}$.

For every $\bar{\mu}$ is expressible in the form:

$$\bar{\mu} = \Sigma(J\hat{\mu}_{0i}\bar{\mu})\bar{\mu}_i + \bar{\mu}_0, \text{ with } J\hat{\mu}_{0i}\bar{\mu}_0 = 0 \text{ for every } i.$$

Now

$$\begin{aligned} J(\kappa_1 + \Sigma\bar{\mu}_i\hat{\mu}_{0i})(\Sigma c_j\bar{\mu}_j) &= \Sigma_jc_jJ\kappa_1\bar{\mu}_j + \Sigma_{ij}c_j\bar{\mu}_iJ\hat{\mu}_{0i}\bar{\mu}_j \\ &= -\Sigma_jc_j\bar{\mu}_j + \Sigma_jc_j\bar{\mu}_j = 0. \end{aligned}$$

Obviously

$$J\bar{\mu}_i\hat{\mu}_{0i}\bar{\mu}_0 = \bar{\mu}_iJ\hat{\mu}_{0i}\bar{\mu}_0 = 0 \text{ for every } i$$

and by (*c*)

$$J\kappa_1\bar{\mu}_0 = 0.$$

Hence

$$J(\kappa_1 + \Sigma\bar{\mu}_i\hat{\mu}_{0i})\bar{\mu} = J(\kappa_1 + \Sigma_i\bar{\mu}_i\hat{\mu}_{0i})(\Sigma_jc_j\bar{\mu}_j + \bar{\mu}_0) = 0.$$

If J is definite on the left it follows that

$$\kappa_1 + \Sigma\bar{\mu}_i\hat{\mu}_{0i} = 0 \qquad \text{or} \qquad \kappa_1 = -\Sigma\bar{\mu}_i\hat{\mu}_{0i}.$$

In other words we have:

(j) $\kappa + \lambda + J\kappa\lambda + \Sigma\bar{\mu}_i\hat{\mu}_{0i} = 0$ with $J\hat{\mu}_{0i}\bar{\mu}_j = \delta_{ij}$.

In order to show the correctness of the second relation:

$$\kappa + \lambda + J\lambda\kappa + \Sigma\bar{\mu}_{0i}\hat{\mu}_i = 0$$

we notice that the equation

$$\lambda + J\kappa\lambda = -\kappa - \Sigma\bar{\mu}_i\hat{\mu}_{0i} = \kappa_1 - \kappa$$

fulfills the conditions required by the hypothesis of the theorem to enable us to solve for λ. We find:

$$\begin{aligned}\lambda &= -\kappa - \Sigma\bar{\mu}_i\hat{\mu}_{0i} - J\lambda(\kappa + \Sigma\bar{\mu}_i\hat{\mu}_{0i}) + \Sigma\bar{\mu}_{0i}\hat{\mu}_{1i}\\ &= -\kappa - J\lambda\kappa - \Sigma(\bar{\mu}_i + J\lambda\bar{\mu}_i)\hat{\mu}_{0i} + \Sigma\bar{\mu}_{0i}\hat{\mu}_{1i}.\end{aligned}$$

By multiplying the first relation by $\bar{\mu}_i$ and operating with J, we find:

$$J\kappa\bar{\mu}_i + J\lambda\bar{\mu}_i + JJ\kappa\lambda\bar{\mu}_i + \bar{\mu}_i = 0,$$

i.e.,

$$\bar{\mu}_i + J\lambda\bar{\mu}_i + J\kappa(\bar{\mu}_i + J\lambda\bar{\mu}_i) = 0$$

and so

$$\bar{\mu}_i + J\lambda\bar{\mu}_i = \Sigma_j c_{ij}\bar{\mu}_{0j}.$$

Hence if we let

$$\hat{\mu}_i = \hat{\mu}_{1i} + \Sigma_j\hat{\mu}_{0j}c_{ji},$$

$$\lambda + \kappa + J\lambda\kappa + \Sigma\bar{\mu}_{0i}\hat{\mu}_i = 0.$$

If we multiply this relation through by $\bar{\mu}_{0j}$ and operate with J, we find:

$$J\lambda\bar{\mu}_{0j} + J\kappa\bar{\mu}_{0j} + JJ\lambda\kappa\bar{\mu}_{0j} = -\Sigma_i\bar{\mu}_{0i}J\hat{\mu}_i\bar{\mu}_{0j}$$

or

$$\bar{\mu}_{0j} = \Sigma_i\bar{\mu}_{0i}J\hat{\mu}_i\bar{\mu}_{0j},$$

and hence on account of the linear independence of the $\bar{\mu}_{0j}$,

$$J\hat{\mu}_i\bar{\mu}_{0j} = \delta_{ij}.$$

This completes the proof of the theorem.

If we define pseudo-resolvents in terms of the solution of the adjoint equation $\hat{\xi} + J\hat{\xi}\kappa = \hat{\eta}$, we get the same result as in Theorem 5, if we assume J to be definite on the right.

Obviously we can show that a pseudo-resolvent λ for the equation $\bar{\xi} + J\kappa\bar{\xi} = \bar{\eta}$ will also be a pseudo-resolvent for the adjoint equation $\hat{\xi} + J\hat{\xi}\kappa = \hat{\eta}$ and conversely.

If we drop the condition that J is definite on the left in Theorem 5 it becomes:

A necessary and sufficient condition that λ be a pseudo-resolvent is that

$$\kappa + \lambda + J\kappa\lambda + \Sigma\bar{\mu}_i\hat{\mu}_{0i} = \kappa_0,$$

$$\kappa + \lambda + J\lambda\kappa + \Sigma\bar{\mu}_{0i}\hat{\mu}_i = \kappa_0 + J\lambda\kappa_0,$$

where $J\hat{\mu}_{0i}\bar{\mu}_j = J\hat{\mu}_i\bar{\mu}_{0j} = \delta_{ij}$, and κ_0 is such that $J\kappa_0\bar{\mu} = 0$ for every $\bar{\mu}$ of the class $\mathfrak{M}$, and $J\hat{\mu}_{0i}\kappa_0 = 0$ for every i. Such a λ will not serve as a pseudo-resolvent for $\hat{\xi} + J\hat{\xi}\kappa = \hat{\eta}$ unless we have also $J\hat{\mu}\kappa_0 = 0$ for every $\hat{\mu}$ of $\hat{\mathfrak{M}}$. Then $J\lambda\kappa_0 = 0$ also.

The following theorem is of importance in obtaining functions λ satisfying the pseudo-reciprocal relations of Theorem 5:

6. *A necessary and sufficient condition that a kernel of the form:*

$$\kappa_1 = \kappa + \Sigma_{ij}a_{ij}\bar{\mu}_i\hat{\mu}_j,$$

where $J\hat{\mu}_{0i}\bar{\mu}_j = J\hat{\mu}_i\bar{\mu}_{0j} = \delta_{ij}$, *have no zero solutions, i.e., be such that* $\bar{\xi} + J\kappa\bar{\xi}_1 = 0$ *has as consequence* $\bar{\xi} = 0$, *is that the determinant of the elements* a_{ij} *be different from zero. The reciprocal* λ *of any such* κ_1 *will satisfy the relations of Theorem 5, i.e., be a pseudo-resolvent of* κ.

Suppose that $\bar{\xi}$ is a solution of

$$\bar{\xi} + J\kappa_1\bar{\xi} = \bar{\xi} + J(\kappa + \Sigma_{ij}a_{ij}\bar{\mu}_i\hat{\mu}_j)\bar{\xi} = 0.$$

If we multiply by $\hat{\mu}_{0i}$ and operate with J, we get:

$$\Sigma_j a_{ij}J\hat{\mu}_j\bar{\xi} = 0,$$

and so

$$\bar{\xi} + J\kappa\bar{\xi} = 0, \qquad \text{i.e.,} \qquad \bar{\xi} = \Sigma_k c_k\bar{\mu}_{0k}.$$

In order to determine the c_k we substitute in the relation $\Sigma_j a_{ij}J\hat{\mu}_j\bar{\xi} = 0$, and obtain:

$$\Sigma_j a_{ij}c_j = 0.$$

A necessary and sufficient condition that $c_j = 0$ be the only solution of this set of equations is that the determinant of the a_{ij} be different from zero.

The second part of the theorem follows as in Theorem 3.

Obviously the determination of λ satisfying the pseudo-reciprocal is not unique. We note that:

7. *If* λ_1 *and* λ_2 *both satisfy the equations of Theorem 5 for the same* κ, *then they differ by a function of the form* $\Sigma_{ij}b_{ij}\bar{\mu}_{0i}\hat{\mu}_{0j}$.

For the difference $\lambda_1 - \lambda_2$ of two such functions will satisfy the equations:

$$\lambda_1 - \lambda_2 + J\kappa(\lambda_1 - \lambda_2) = 0,$$

$$\lambda_1 - \lambda_2 + J(\lambda_1 - \lambda_2)\kappa = 0,$$

from which it follows that $\lambda_1 - \lambda_2$ is expressible in the form $\Sigma_{ij}b_{ij}\bar{\mu}_{0i}\hat{\mu}_{0j}$.

In the above two theorems it was assumed that $\bar{\mu}_{0i}$ and $\hat{\mu}_{0i}$ constituted complete sets of solutions of the homogeneous equations $\bar{\xi} + J\kappa\bar{\xi} = 0$ and $\hat{\xi} + J\hat{\xi}\kappa = 0$, respectively. If we drop this condition, we get:

8. If κ and λ satisfy a pair of equations of the form of the pseudo-reciprocal relations of Theorem 5:

$$\kappa + \lambda + J\kappa\lambda + \Sigma\bar{\mu}_i\hat{\mu}_{0i} = 0,$$
$$\kappa + \lambda + J\lambda\kappa + \Sigma\bar{\mu}_{0i}\hat{\mu}_i = 0,$$

in which $J\hat{\mu}_i\bar{\mu}_{0j} = J\hat{\mu}_{0i}\bar{\mu}_j = \delta_{ij}$ $(i, j = 1, \cdots, n)$ so that each of the four sets of functions $\bar{\mu}_i$, $\hat{\mu}_i$, $\bar{\mu}_{0i}$, $\hat{\mu}_{0i}$ constitutes a linearly independent set, then the solutions of the equation $\bar{\xi} + J\kappa\bar{\xi} = 0$ are linear combinations of $\bar{\mu}_{0i}$; of the equation $\hat{\xi} + J\hat{\xi}\kappa = 0$, are linear combinations of $\hat{\mu}_{0i}$; and obviously on account of the symmetry, the solutions of the corresponding equations in the λ will be linear combinations of the $\bar{\mu}_i$ and $\hat{\mu}_i$, respectively.

For suppose $\bar{\mu} + J\kappa\bar{\mu} = 0$. Then if we multiply the second of the relations of the theorem by $\bar{\mu}$ and operate with J we find:

$$J\kappa\bar{\mu} + J\lambda\bar{\mu} + JJ\lambda\kappa\bar{\mu} + \Sigma\bar{\mu}_{0i}J\hat{\mu}_i\bar{\mu} = -\bar{\mu} + \Sigma\bar{\mu}_{0i}J\hat{\mu}_i\bar{\mu} = 0$$

or

$$\bar{\mu} = \Sigma(J\hat{\mu}_i\bar{\mu})\bar{\mu}_{0i}.$$

The other results are obtained in a similar way.

If we multiply the first of the two relations of this theorem by $\hat{\mu}_{0i}$ on the right and operate with J we get:

$$J\hat{\mu}_{0i}\kappa + J\hat{\mu}_{0i}\lambda + JJ\mu_{0i}\kappa\lambda + \hat{\mu}_{0i} = 0,$$

i.e., $\hat{\mu}_{0i} + J\hat{\mu}_{0i}\kappa$ is a solution of the equation $\hat{\xi} + J\hat{\xi}\lambda = 0$. Similar results are obtainable for the other three homogeneous equations.

9. *If κ and λ satisfy the relations of Theorem 8, and if the equation $\bar{\xi} + J\lambda\bar{\xi} = 0$ has no solution excepting $\bar{\xi} = 0$, then the $\bar{\mu}_{0i}$ and $\hat{\mu}_{0i}$ are the solutions of $\bar{\xi} + J\kappa\bar{\xi} = 0$ and $\hat{\xi} + J\hat{\xi}\kappa = 0$ respectively; and λ is a reciprocal of a function of the form:*

$$\kappa_1 = \kappa + \Sigma_{ij}a_{ij}\bar{\mu}_i\hat{\mu}_j.$$

The first part of the theorem follows immediately from the remarks at the close of Theorem 8. As for the second part, suppose that κ_1 is a reciprocal of λ. Then

$$\kappa_1 + \lambda + J\kappa_1\lambda = 0.$$

Subtracting this from the first of the pseudo-reciprocal relations, we get:

$$\kappa - \kappa_1 + J(\kappa - \kappa_1)\lambda + \Sigma\bar{\mu}_i\hat{\mu}_{0i}.$$

Since κ_1 is a reciprocal of λ we have:

$$\kappa - \kappa_1 = -\Sigma\bar{\mu}_i(\hat{\mu}_{0i} + J\hat{\mu}_{0i}\kappa_1).$$

In a similar way we show that

$$\kappa - \kappa_1 = -\Sigma(\bar{\mu}_{0i} + J\kappa_1\bar{\mu}_{0i})\hat{\mu}_i,$$

and so

$$\kappa - \kappa_1 = -\Sigma_{ij}a_{ij}\bar{\mu}_i\hat{\mu}_j.$$

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